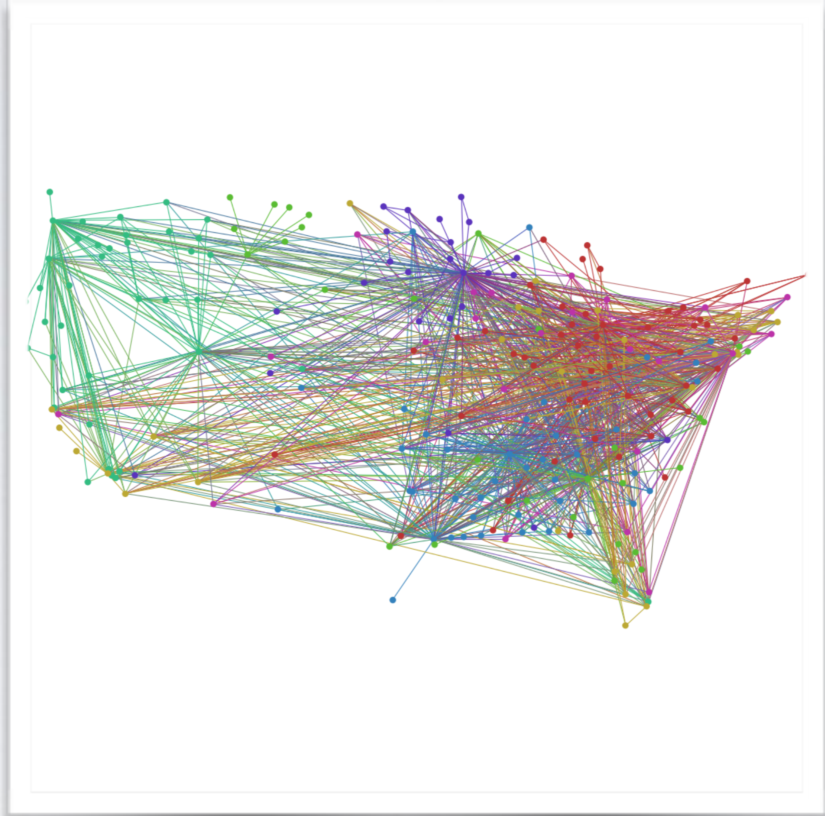
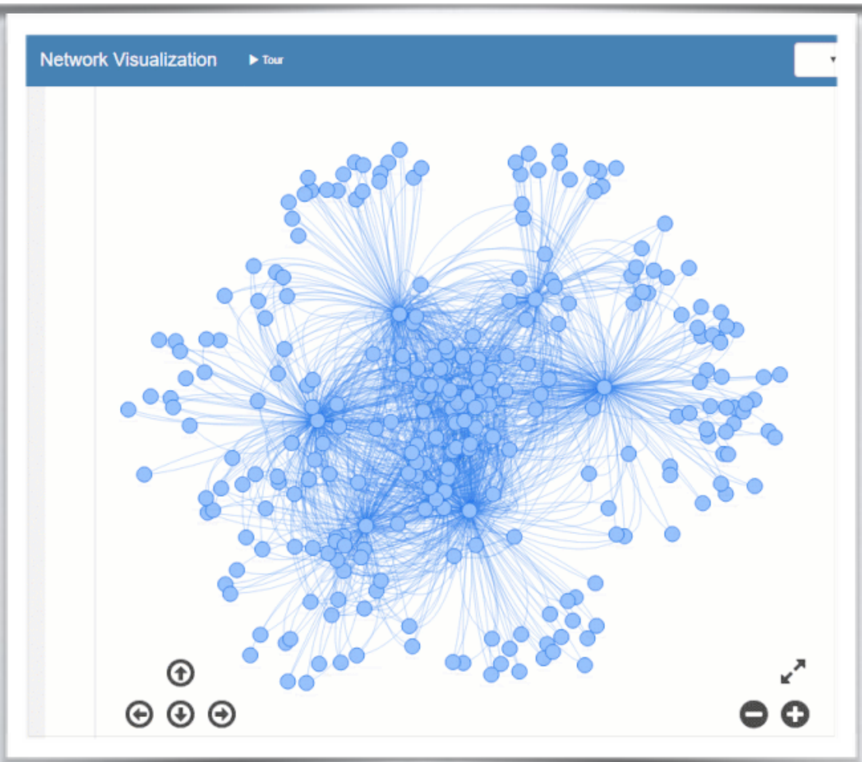
NETWORK VISUALISATION (SHORT DIGRESSION)

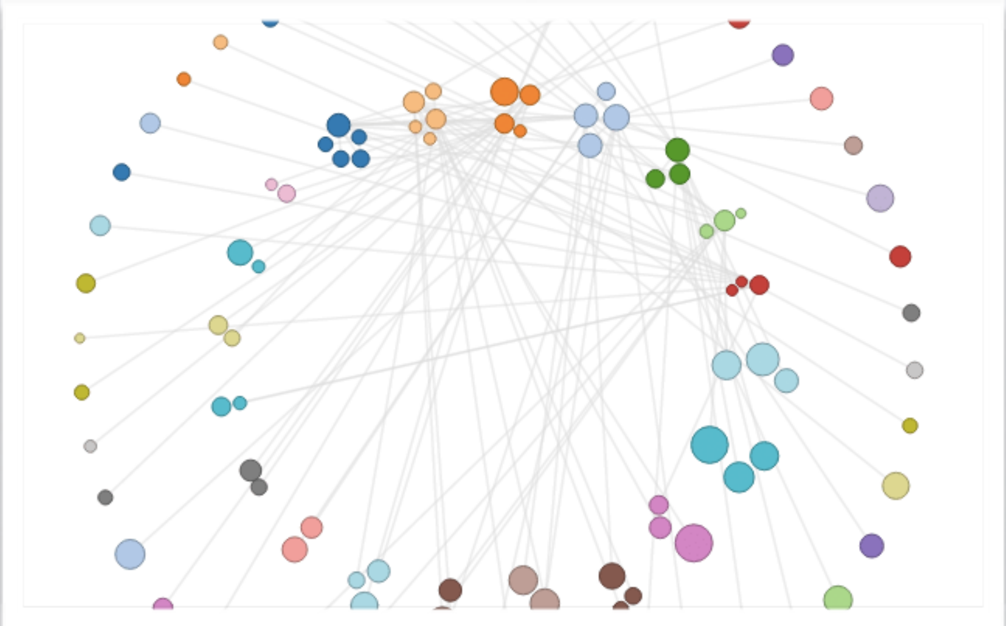
NETWORK VISUALIZATION

- How to interpret a network drawing?
- What does the position of nodes means?
- Can we draw conclusion from the drawing alone?

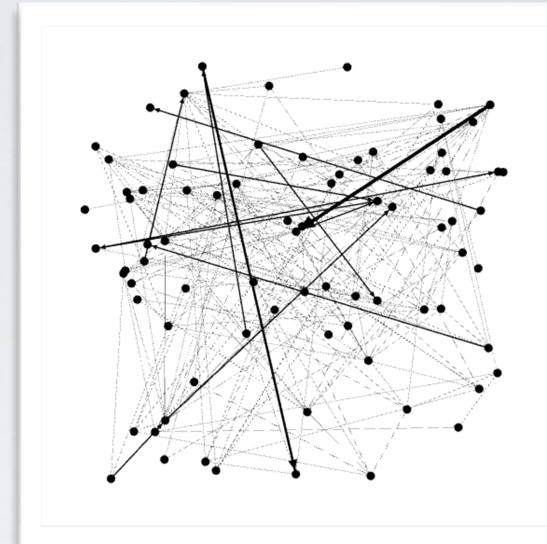
NETWORK VISUALIZATION



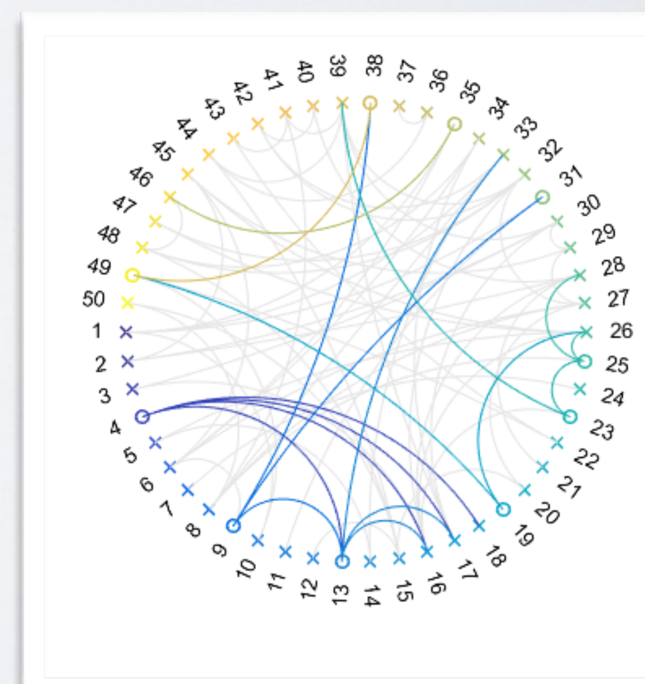




NETWORK VISUALIZATION



- Random layout
  - Assign random positions to nodes, draw edges
    - Useless for more than 5-6 nodes
- Geographical layout
  - The position of nodes is fixed apriori, often based on geographical location
  - Variant: position nodes on a circle based on a single, 1D property (age…)



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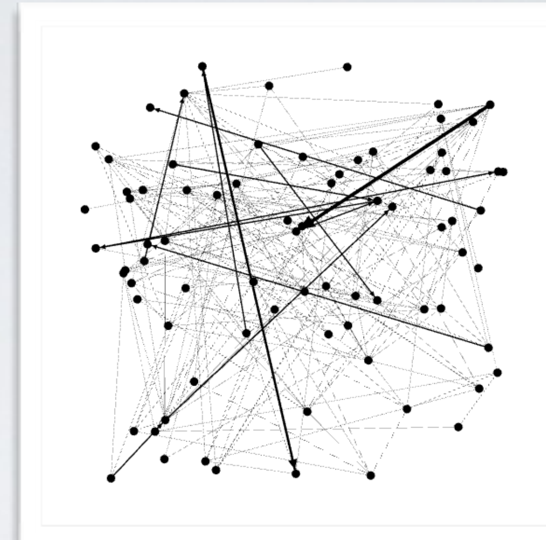

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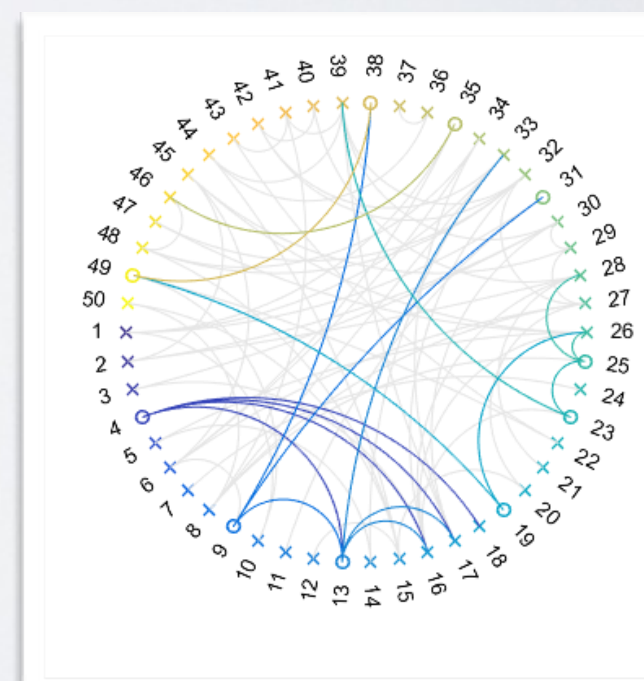
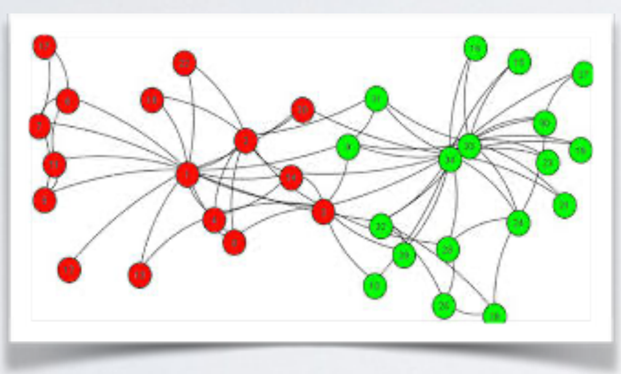
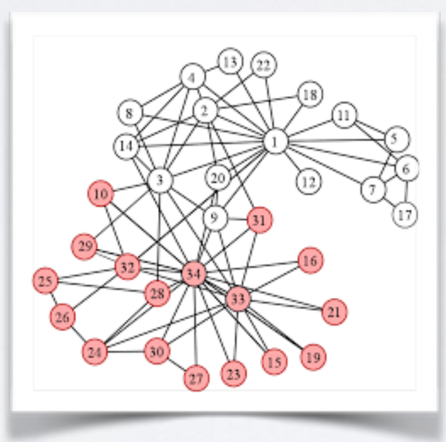
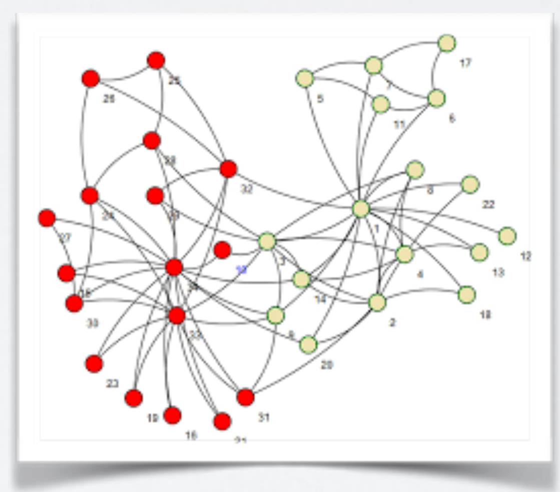
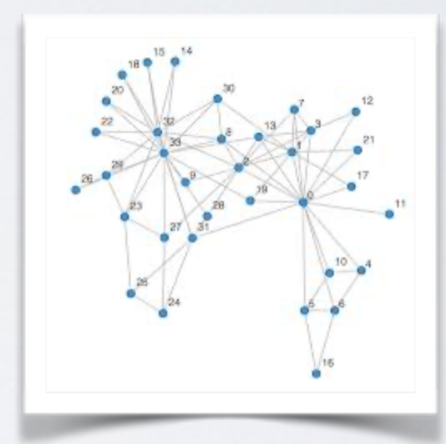
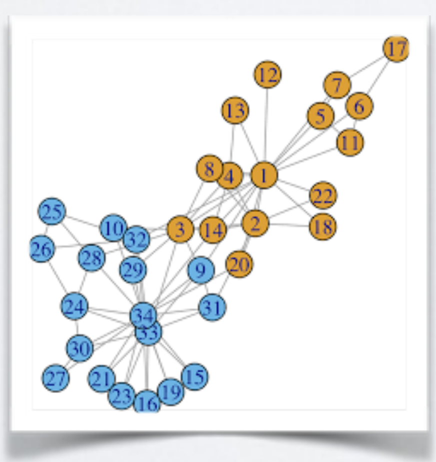


NETWORK VISUALIZATION

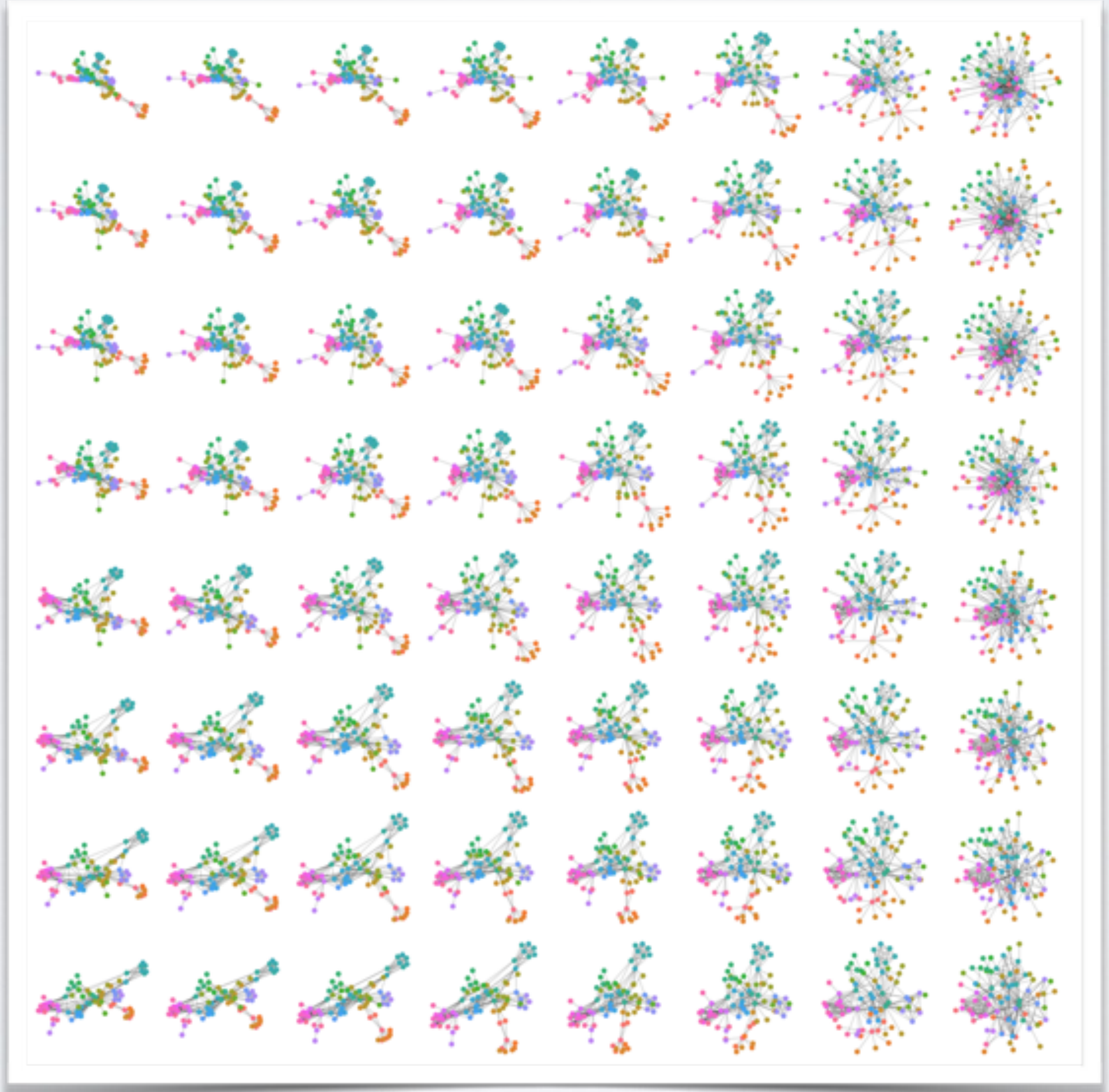
- Most commonly used: Automatic layout
  - Non deterministic
  - Tries to arrange nodes so that the network is easy to read and understand
    - Minimize edge crossings?
    - Most commonly, tries to put connected nodes close and unconnected nodes far



NETWORK VISUALIZATION

http://kwonoh.net/dgl/

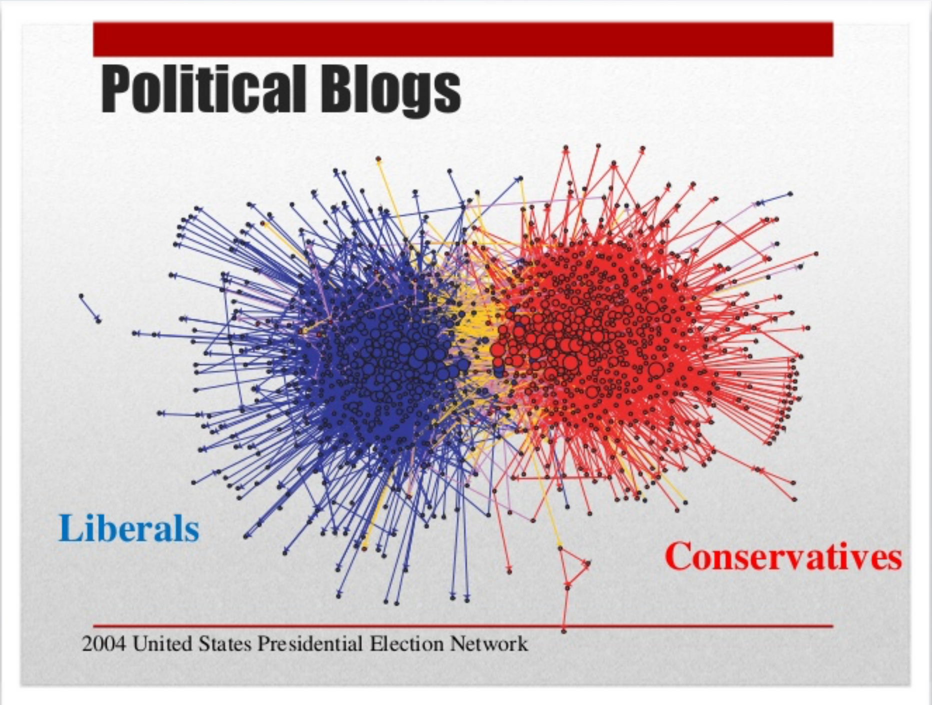


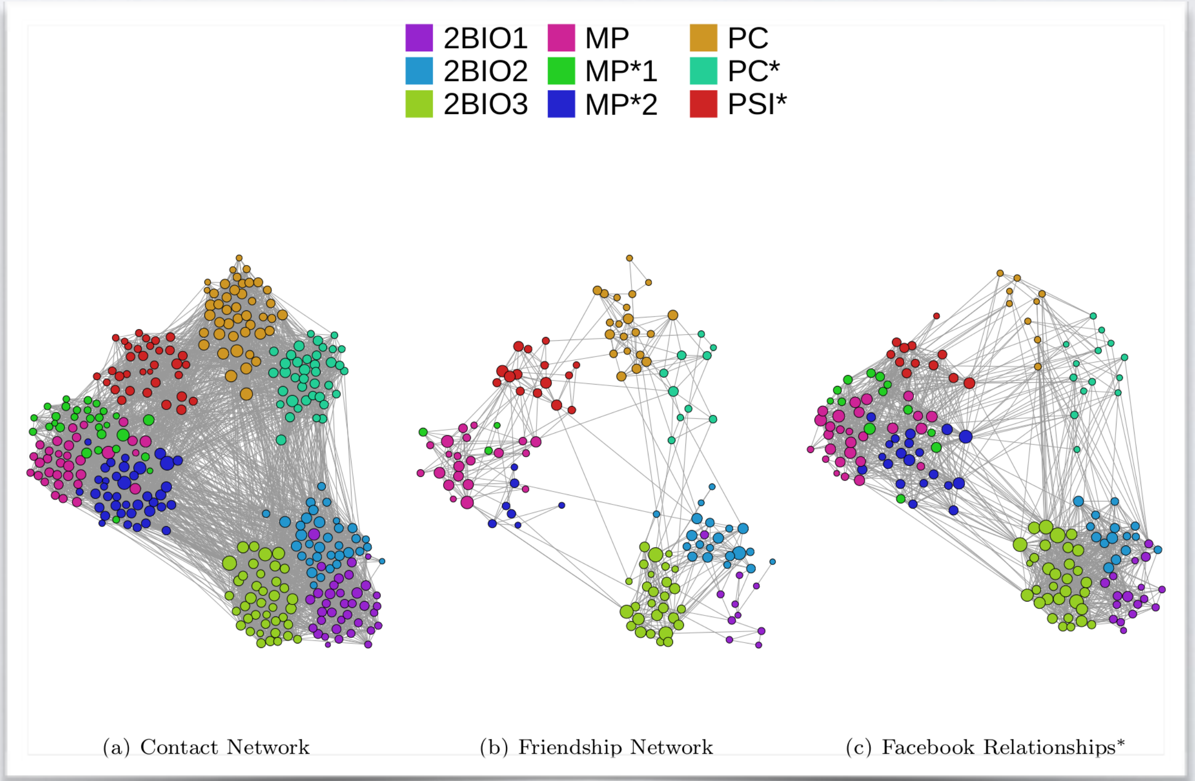
NETWORK VISUALIZATION

- Most common algorithms are variant of the **force directed** layout
  - Kamada-Kawai
  - Fruchterman-Reingold
  - …
- Force directed layout: a simple physical model
  - Repulsive forces between nodes
  - Edges are attracting forces
  - There are minimal (to avoid node overlap) and maximal (to avoid connected component drifting out of the figure)

NETWORK VISUALIZATION

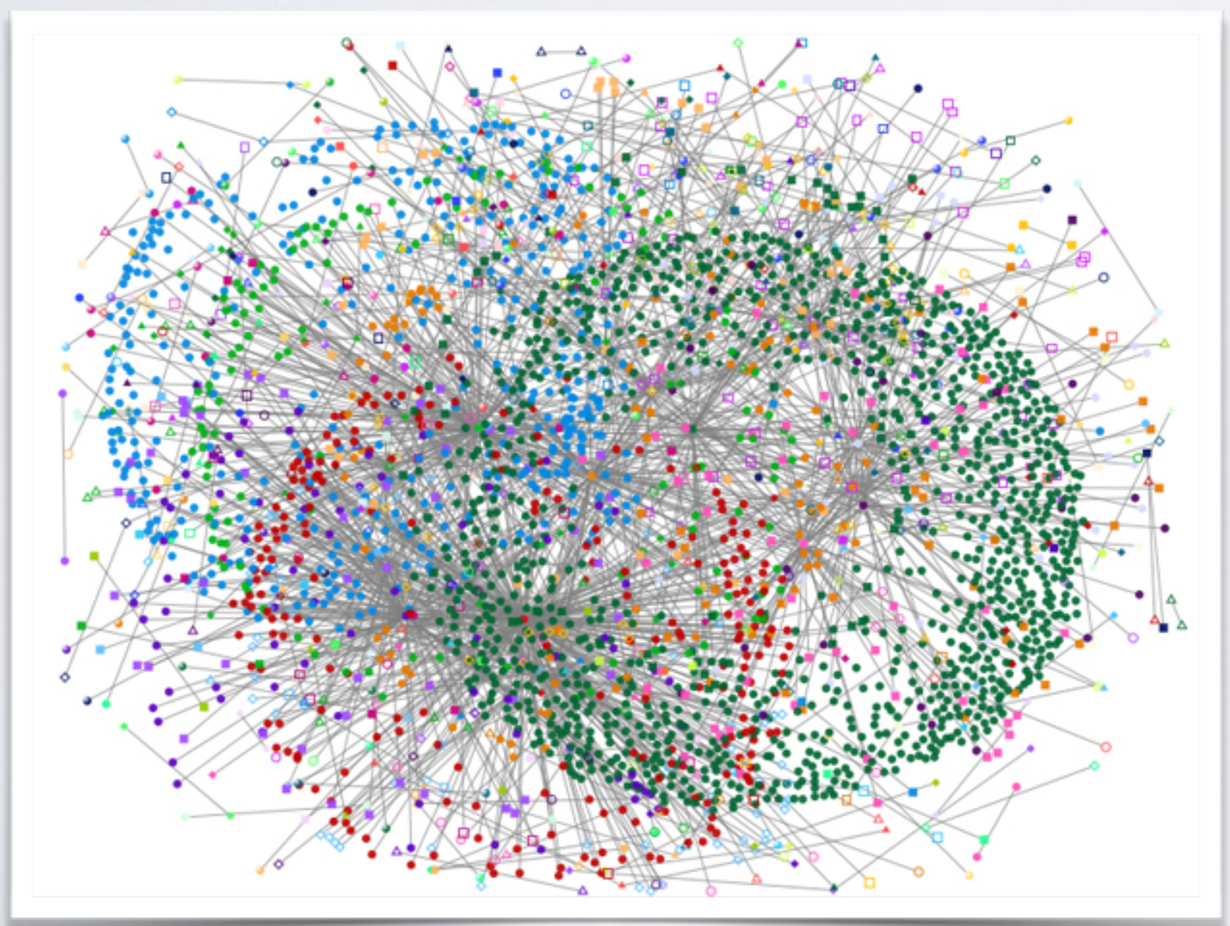
- Can we interpret a force layout?
  - Yes…

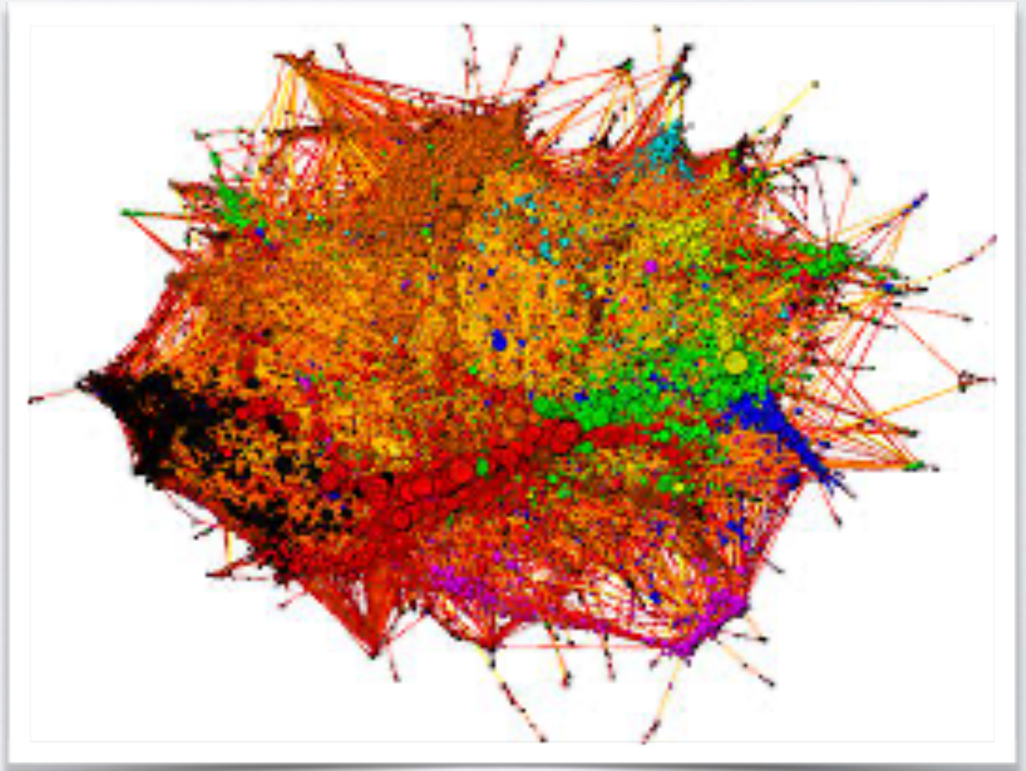




NETWORK VISUALIZATION

- Can we interpret a force layout?
  - Yes…
  - And no.





NETWORK VISUALIZATION

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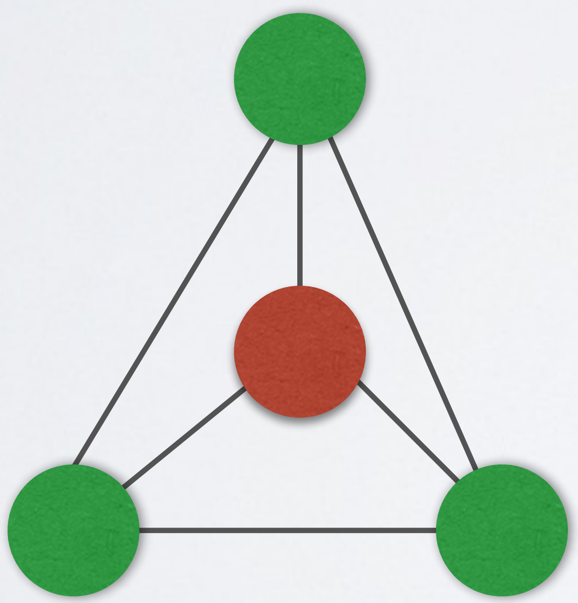
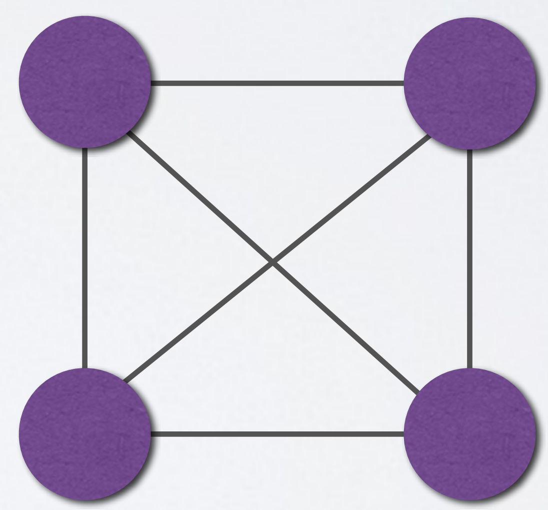




Scale-free networks

Scale-free networks

A network is called ***Scale-free*** **when its degree distribution follows (to some extent) a Power-law distribution**

Power-law distribution: (PDF)

$$P(k) \sim Ck^{-\alpha} = C\frac{1}{k^{\alpha}}$$

α (sometimes γ) called the **exponent** of the distribution

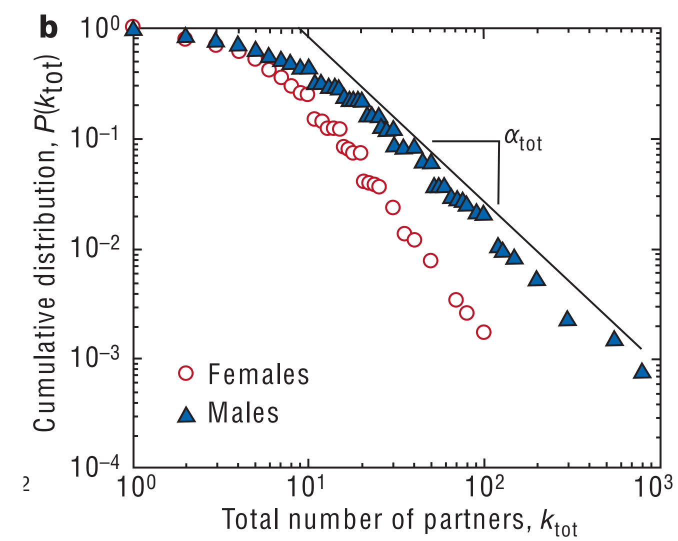
Positive values

Here, defined as continuous (approximation)

Proper definition

Initial definition: $$P(k) \sim Ck^{-\alpha} = C\frac{1}{k^{\alpha}}$$

To have a proper degree distribution, we need
$$\int P(k) = 1 = \int Ck^{-\alpha} = C\int k^{-\alpha}.$$



We also know that in most cases, there is a *lower bound* from which the law holds. ($k_{\min}$)

From this, we define the normalisation constant:

$$C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\alpha} dk} = (\alpha - 1)k_{\min}^{\alpha-1}$$

Proper definition

$$P(k) \sim Ck^{-\alpha}$$

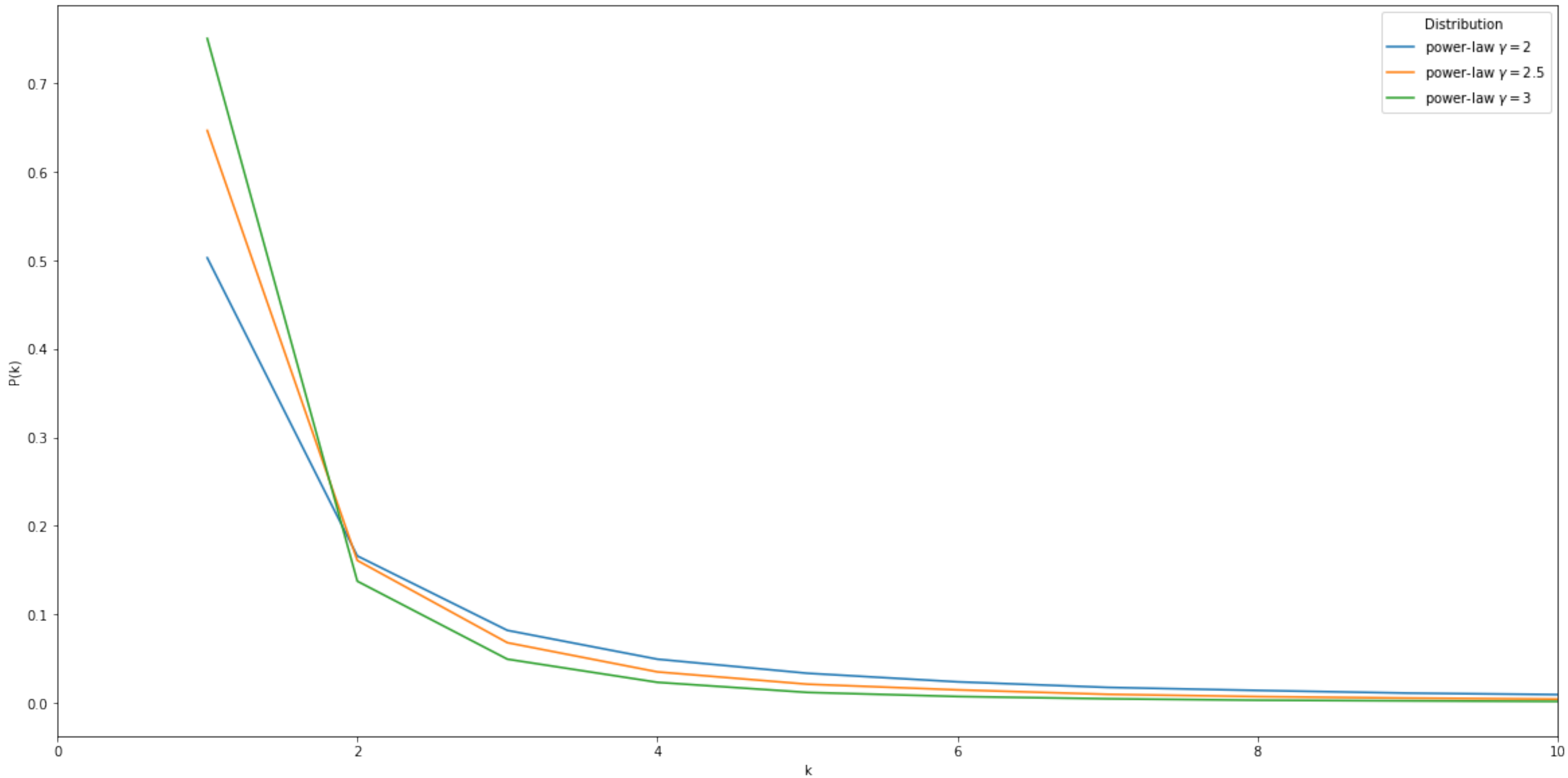
$$C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\alpha} dk} = (\alpha - 1)k_{\min}^{\alpha-1}$$

$$\boxed{\begin{aligned} P(k) &= (\alpha - 1)k_{\min}^{\alpha-1}k^{-\alpha} \\ P(k) &= \frac{\alpha - 1}{k_{\min}}\left(\frac{k}{k_{\min}}\right)^{-\alpha} \end{aligned}}$$

Scale-free networks

Power law plotted with a linear scale, for k<=10

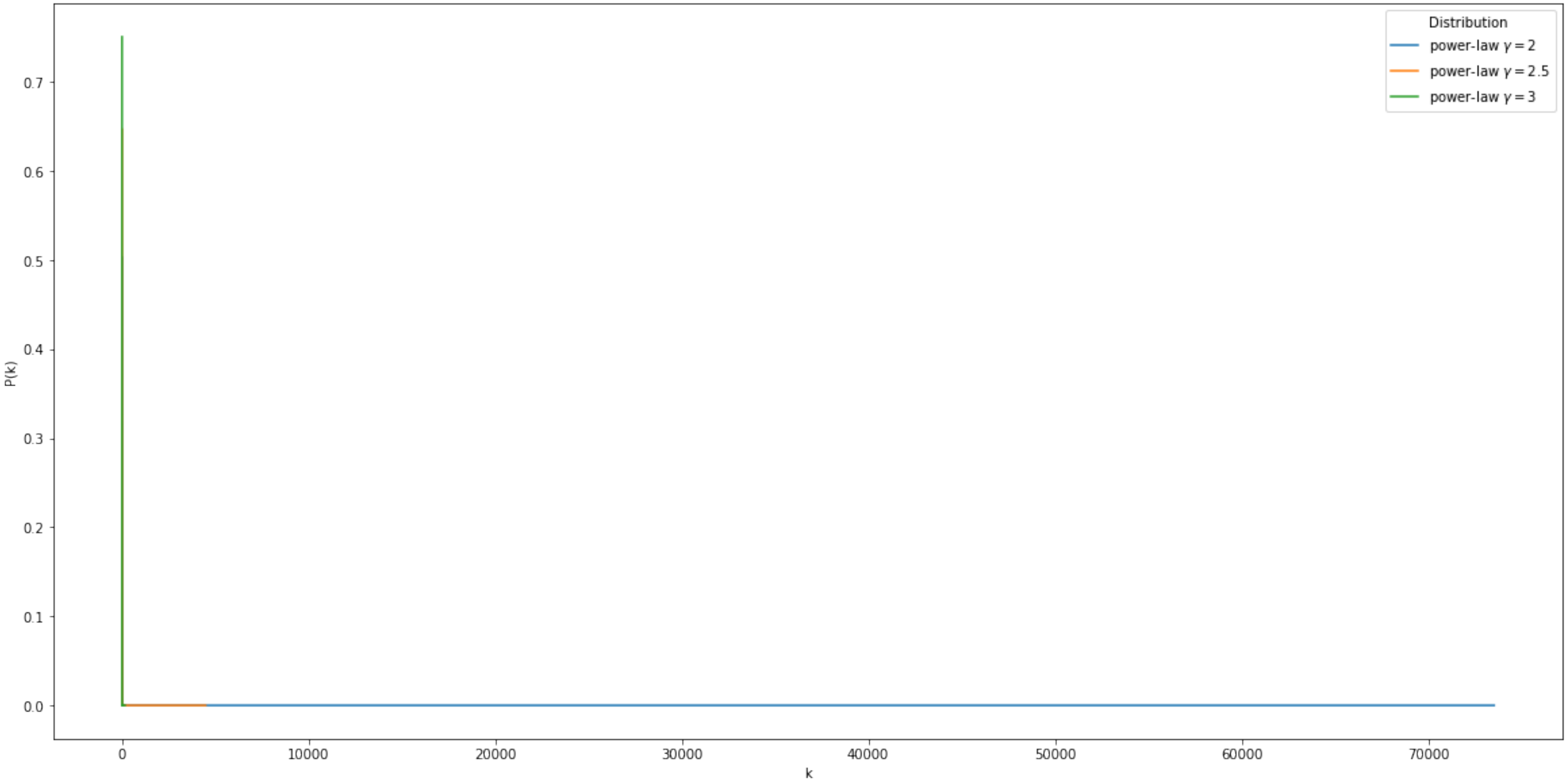
(100 000 samples)



Scale-free networks

Power law plotted with a linear scale, for k<100000

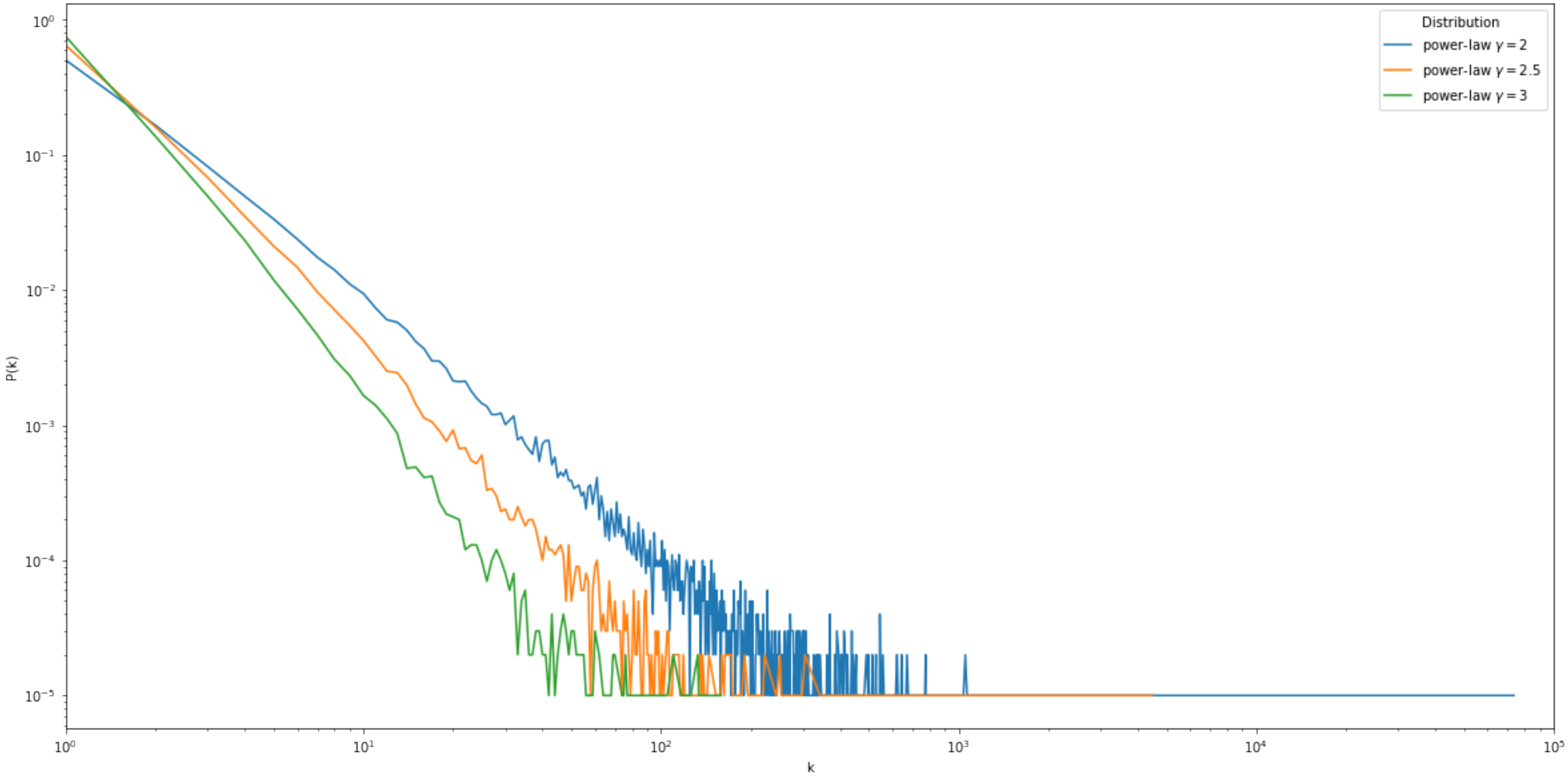
(100 000 samples)



Scale-free networks

Power law plotted with a log-log scale, for k<100000

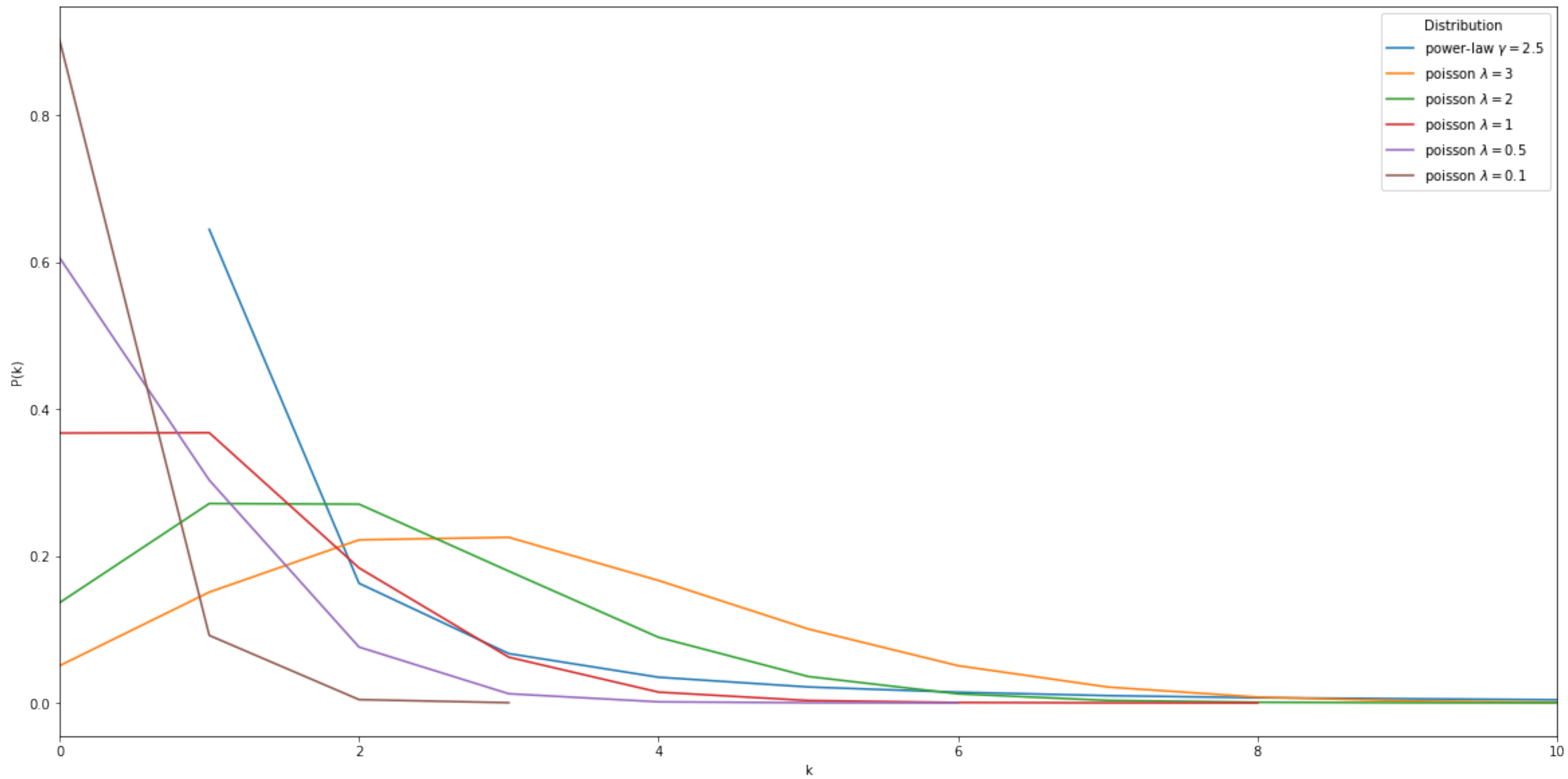
(100 000 samples)



Scale-free networks

Comparing a poisson distribution and a power law

$$\frac{\lambda^k e^{-\lambda}}{k!}$$



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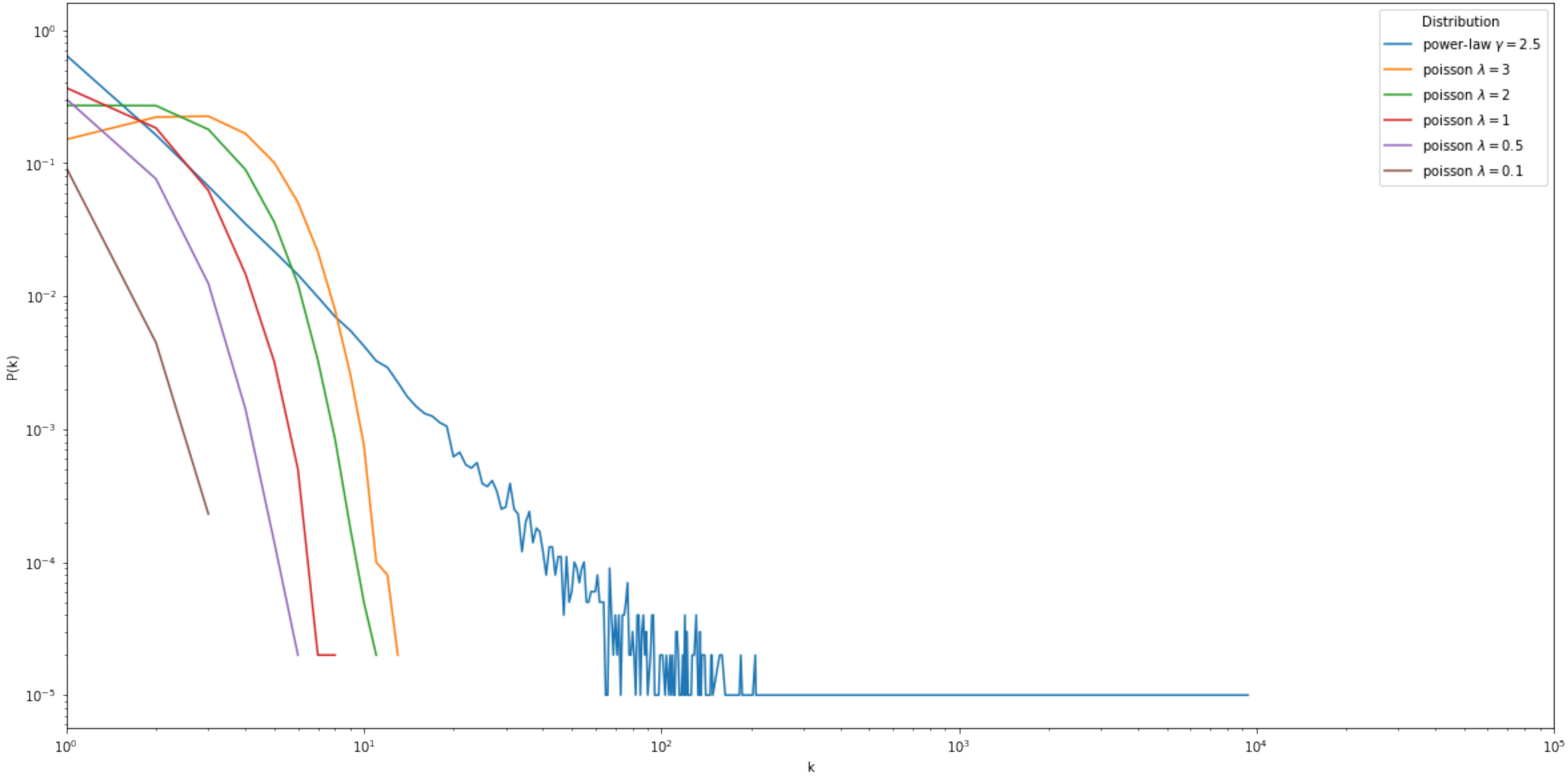

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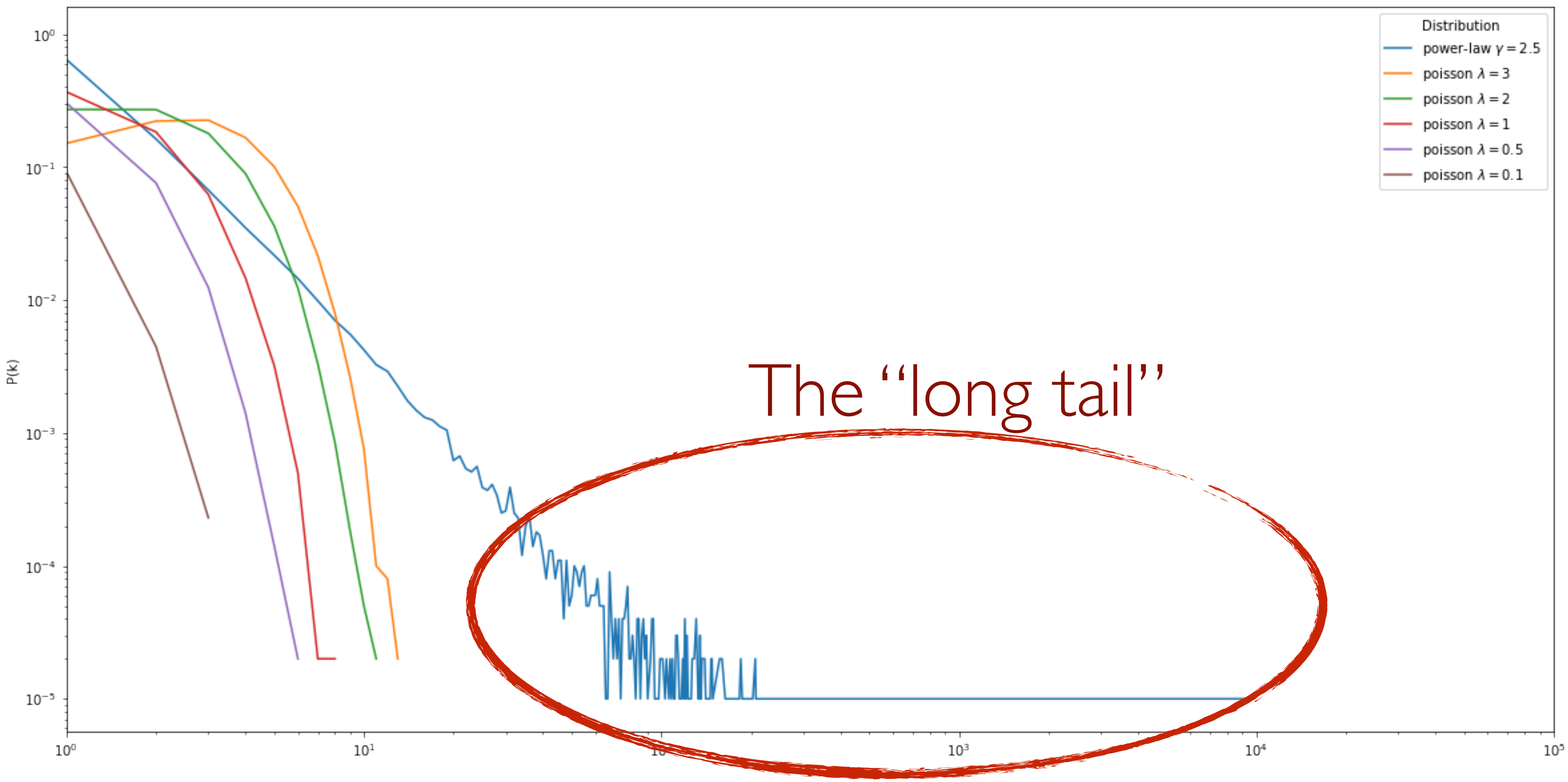
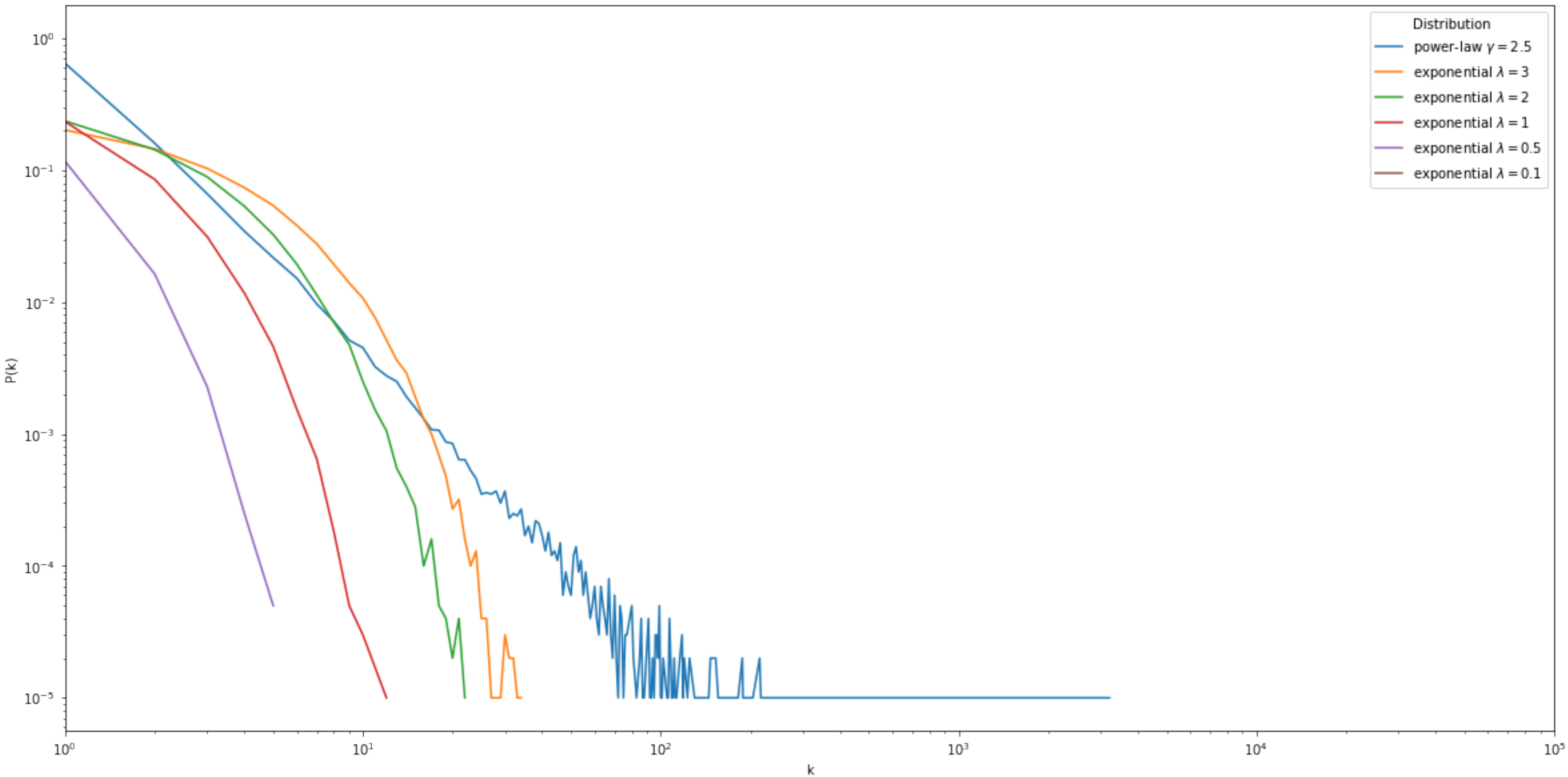


Scale-free networks

Comparing an exponential distribution and a power law

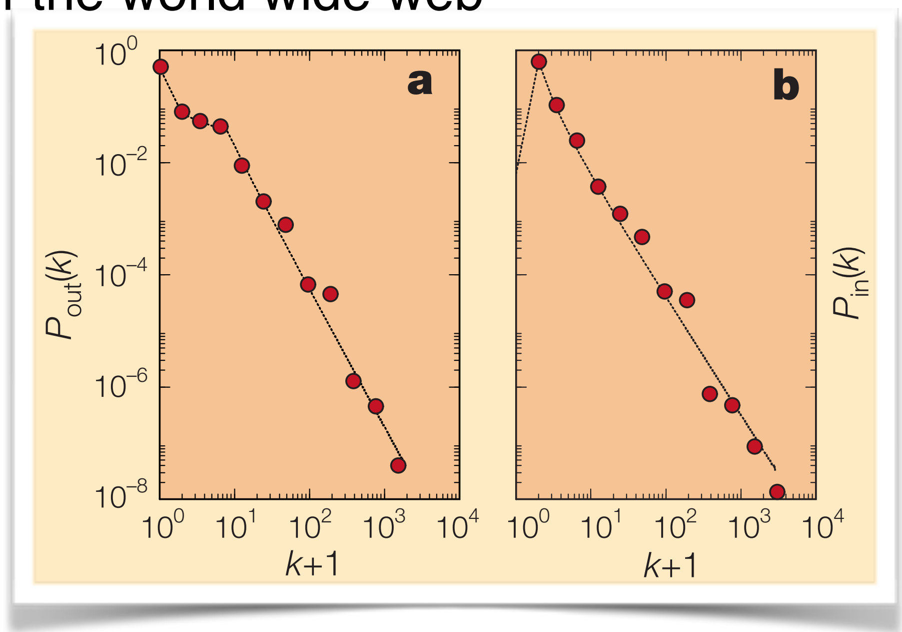
$$\begin{cases} \lambda e^{-\lambda k} & k \geq 0, \\ 0 & k < 0. \end{cases}$$



Scale-free networks - first observations

R. Albert, H. Jeong, A-L Barabási, Nature (1999)

Diameter of the world wide web



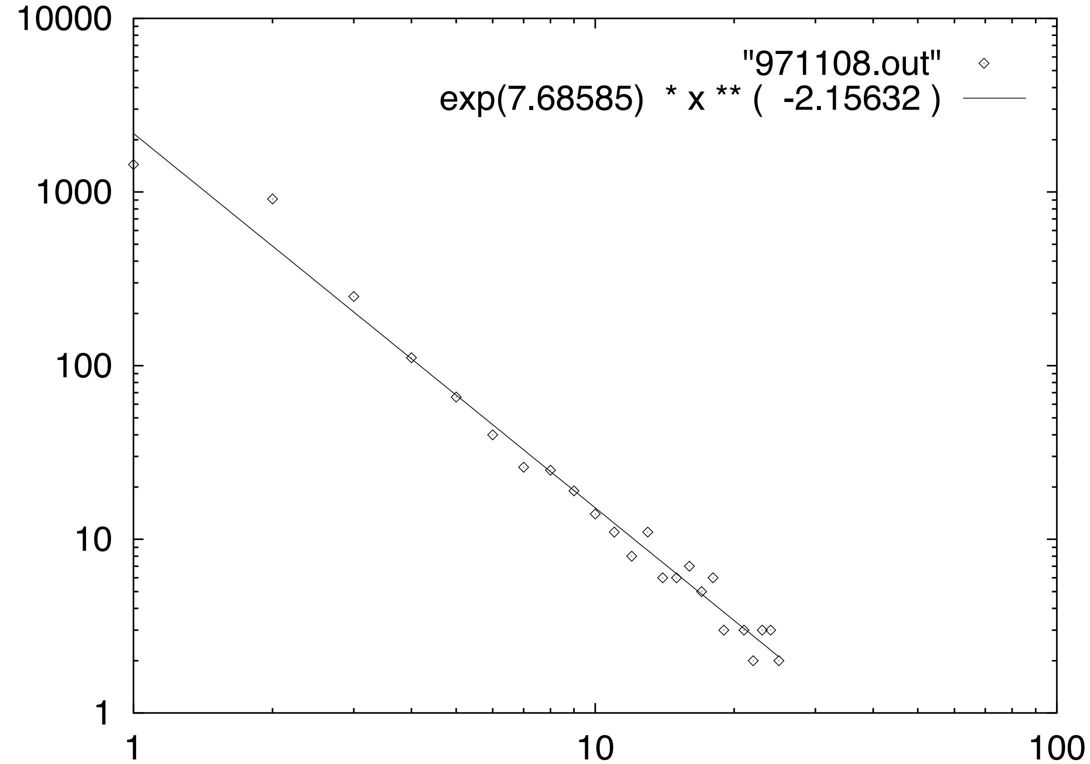

Emergence of Scaling in Random Networks

Albert-László Barabási* and Réka Albert

Scale-free networks - other examples

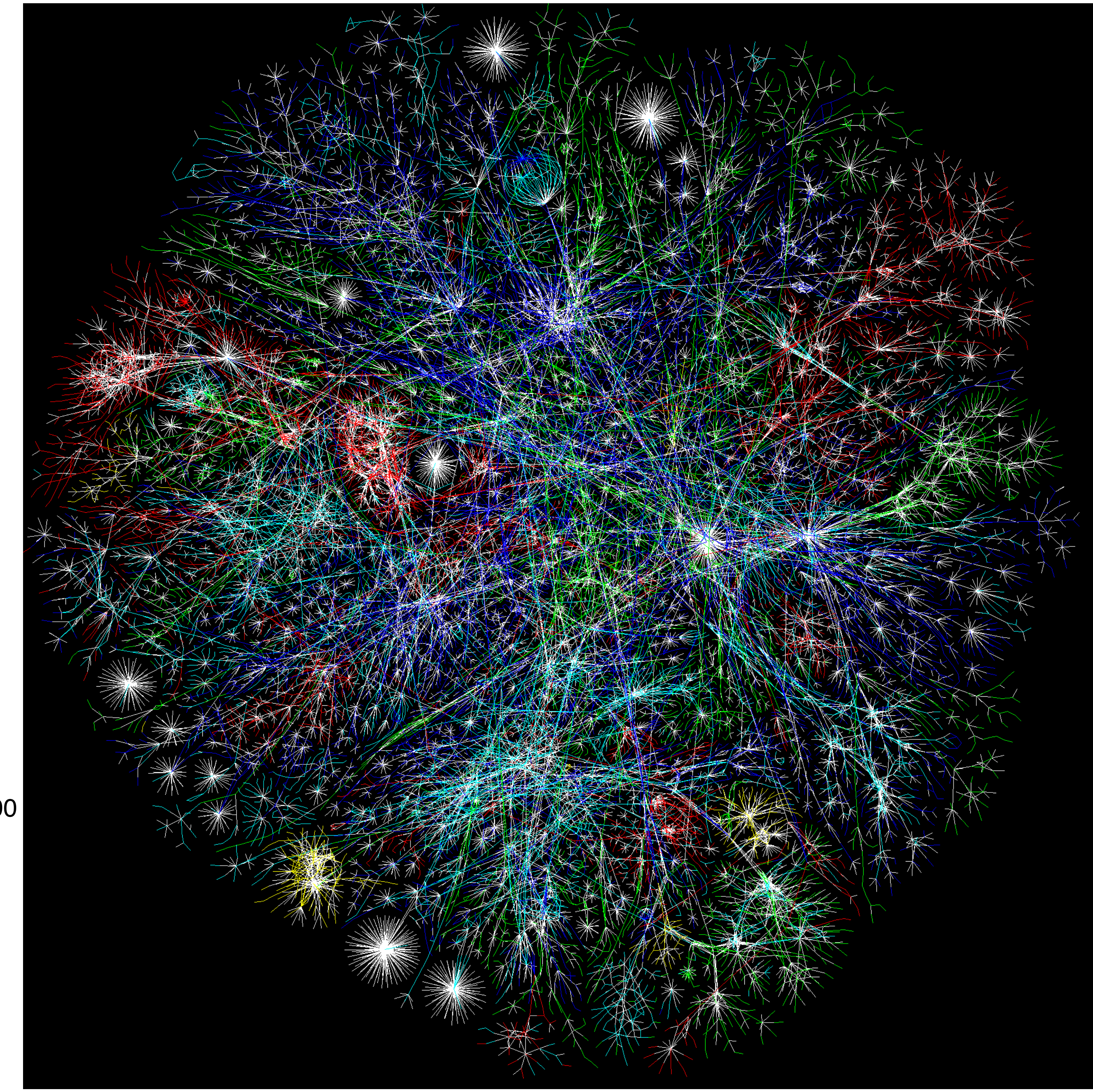
The internet

- Nodes: routers
- Links: Physical wires



(a) Int-11-97

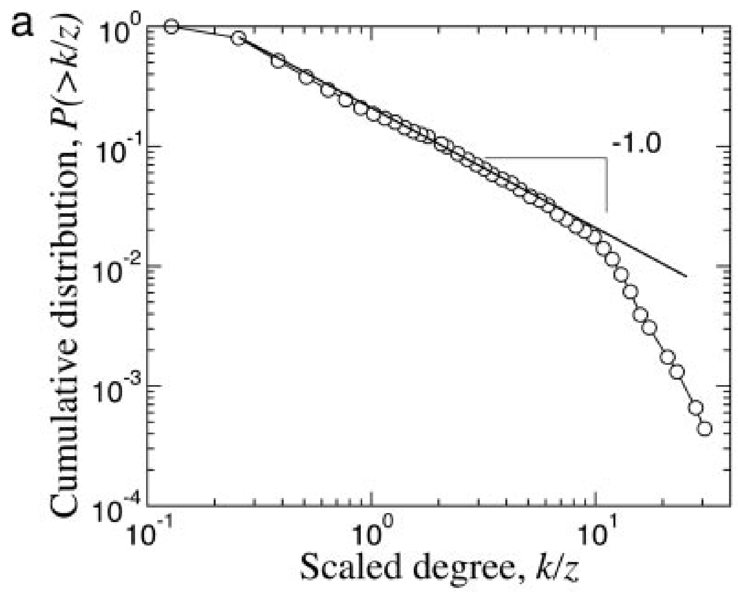
Faloutsos, Faloutsos and Faloutsos (1999)



Scale-free networks - other examples

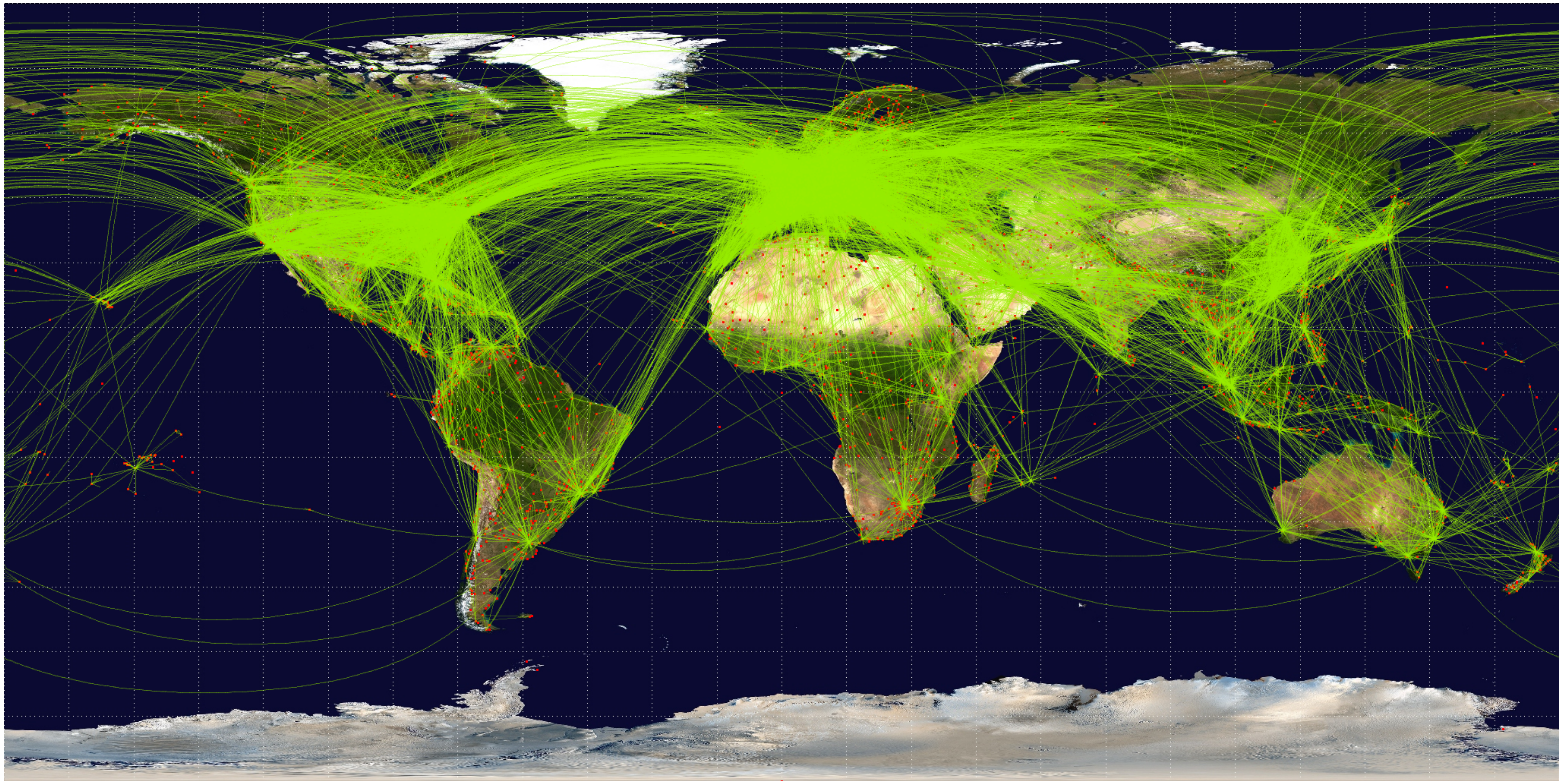
Airline route map network

- Nodes: airports
- Links: airplane connections



Guimera et.al. (2004)

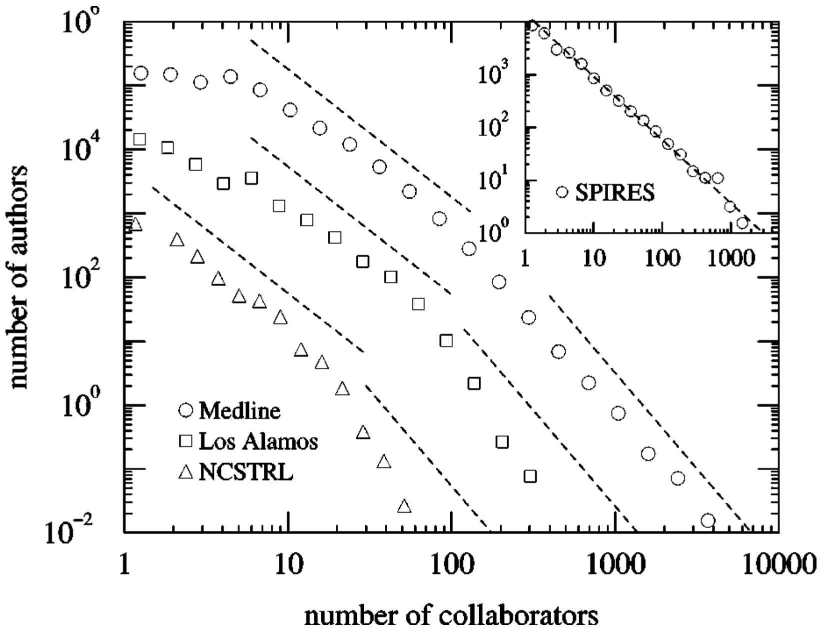
Note: the cumulative distribution of a power law is also a line on a log-log plot



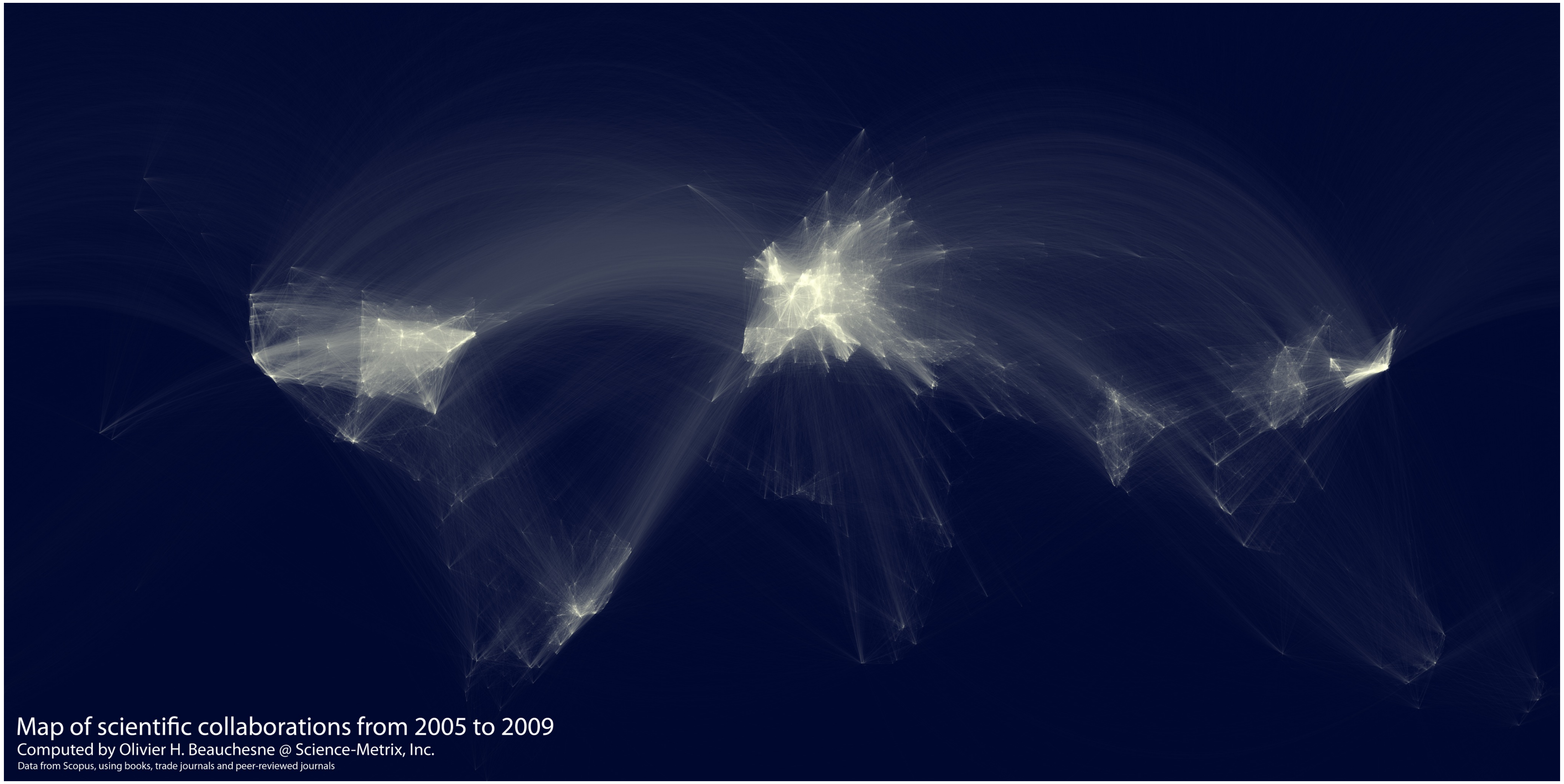
Scale-free networks - other examples

Scientific collaborations

- Nodes: scientists (here geo-localised)
- Links: common papers



Newman (2001)

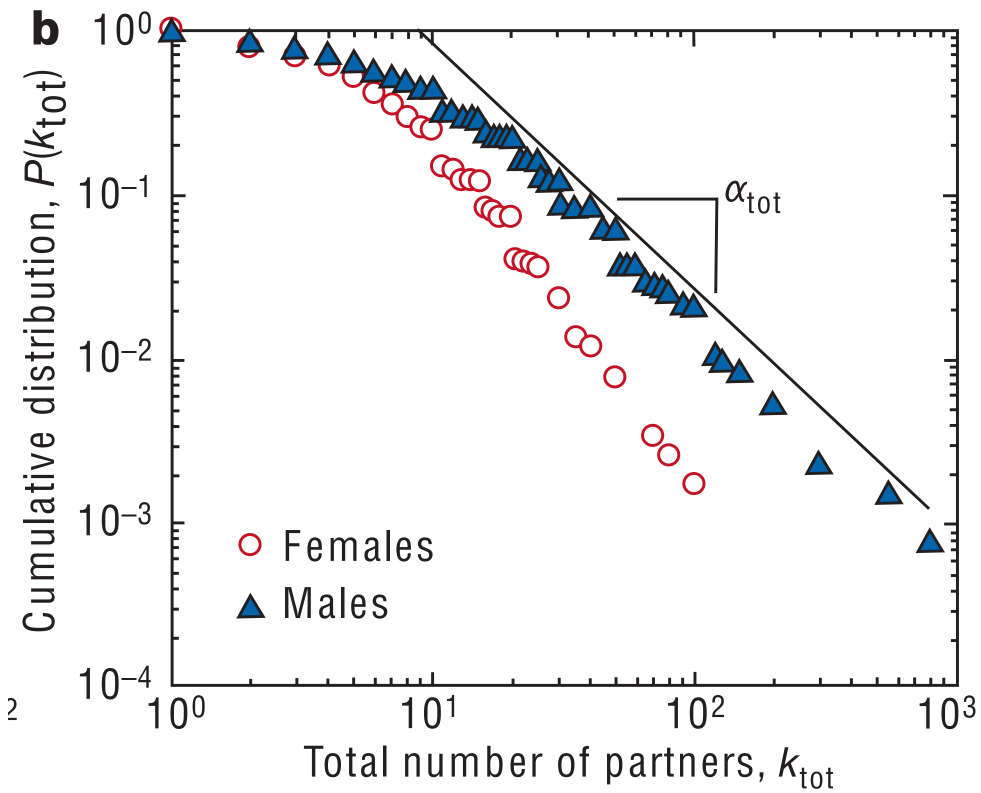


Scale-free networks - other examples

Sexual-interaction networks

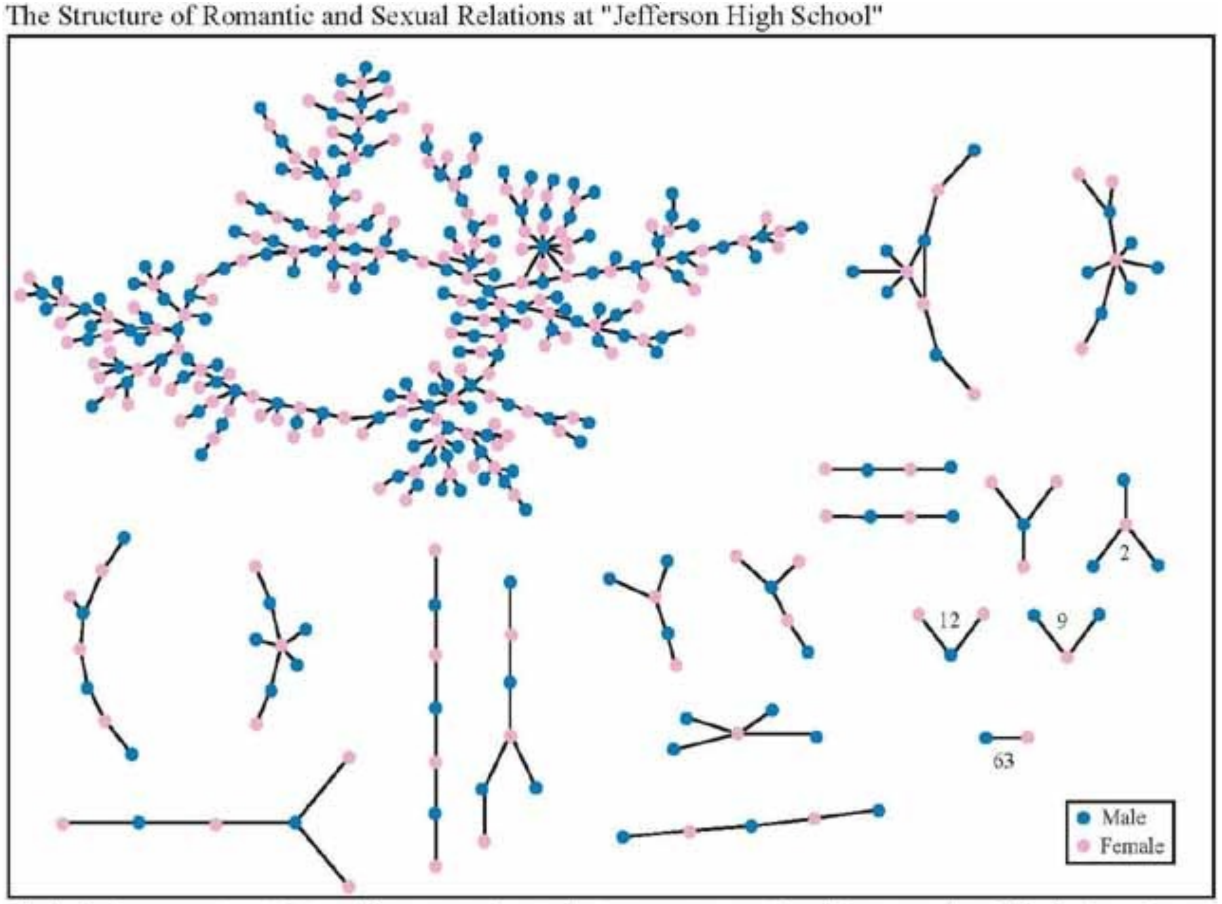
- Nodes: individuals
- Links: sexual incursion

Liljeros et.al. (2001)



Bearman et.al. (2004)

The Structure of Romantic and Sexual Relations at "Jefferson High School"

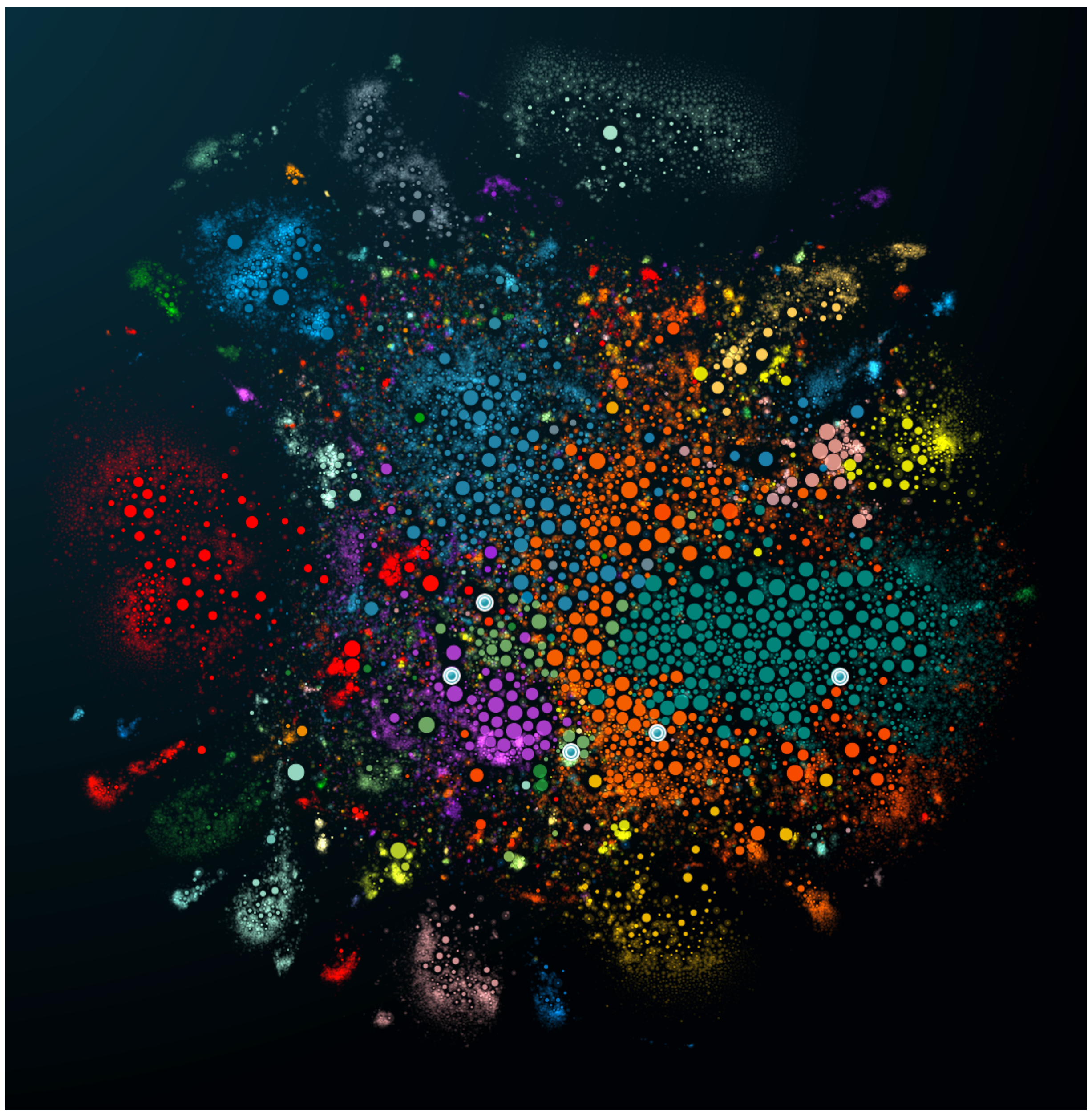


Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

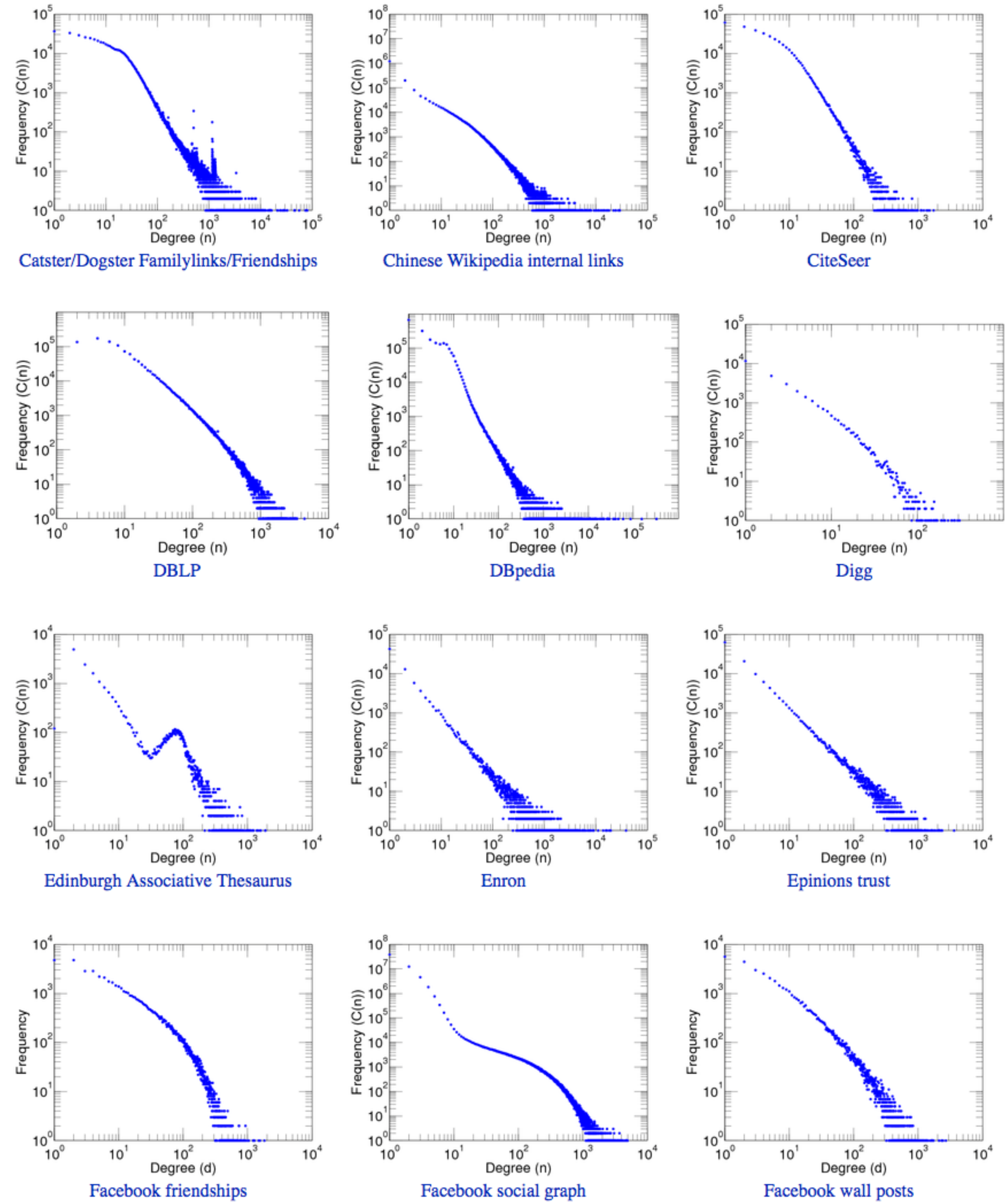
Scale-free networks - other examples

Online social networks

- Nodes: individuals
- Links: online interactions

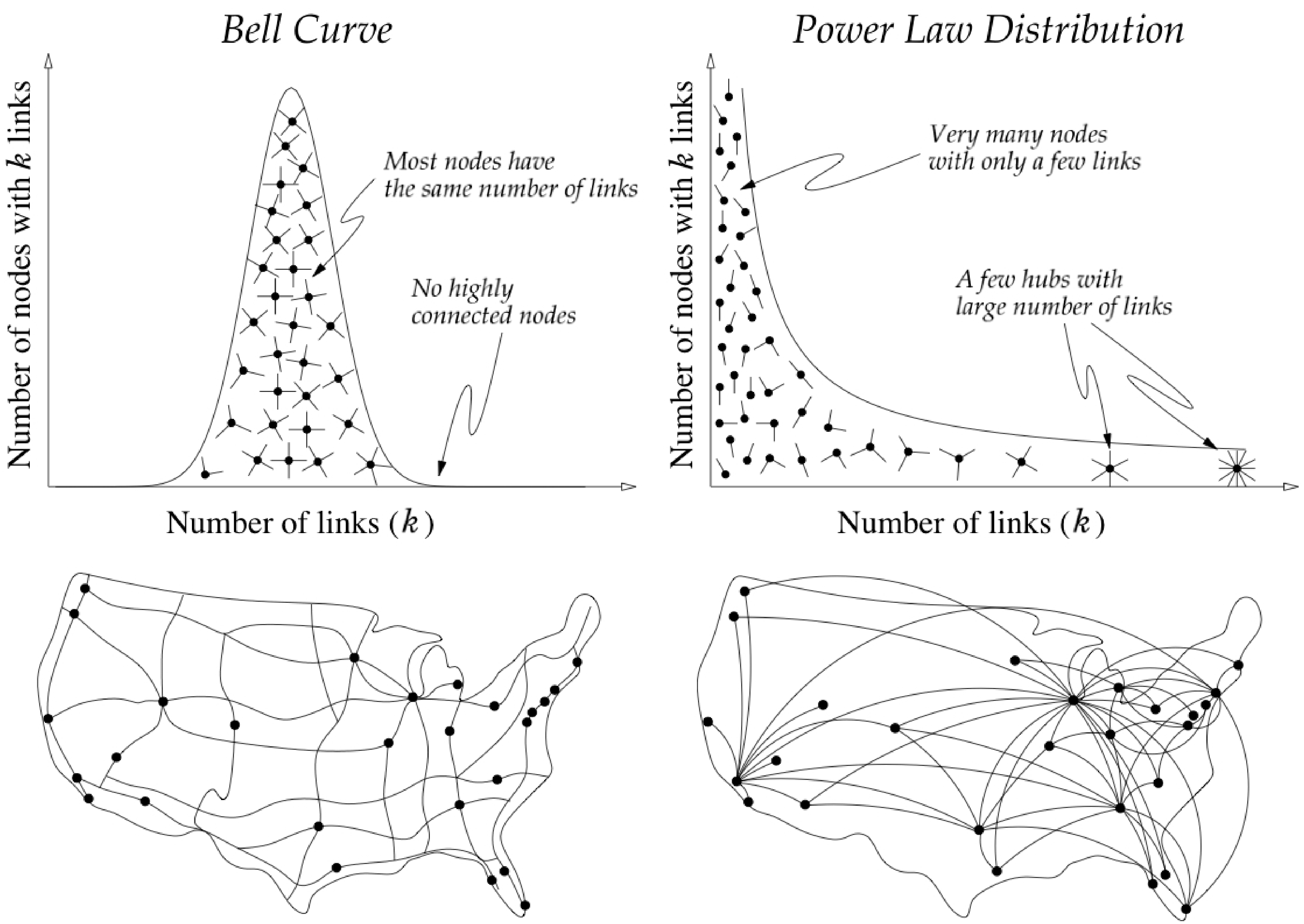


Social network of Steam
http://85.25.226.110/mapper



Scale-free distribution

What does it mean?

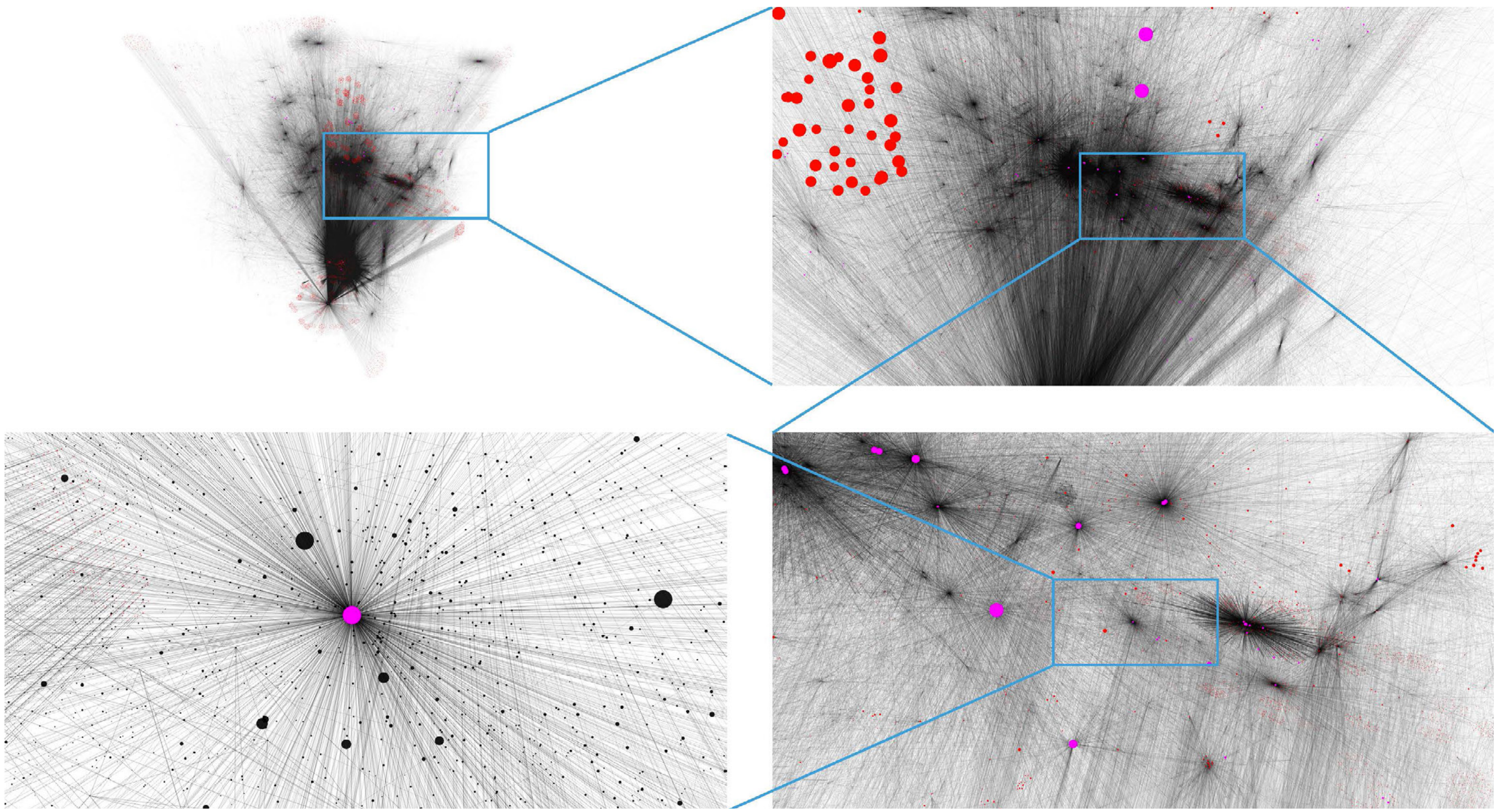


AL. Barabási, Linked (2002)

Degree fluctuations have no characteristic scale (scale invariant)

Scale-free networks

Idea of *scale free*



AL Barabási, Network Science Book (2013)

Interesting properties of power law distributions

$$P(k) \sim Ck^{-\alpha}$$

$$C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\alpha} dk} = (\alpha - 1)k_{\min}^{\alpha-1}$$

$$\boxed{\begin{aligned} P(k) &= (\alpha - 1)k_{\min}^{\alpha-1}k^{-\alpha} \\ P(k) &= \frac{\alpha - 1}{k_{\min}}\left(\frac{k}{k_{\min}}\right)^{-\alpha} \end{aligned}}$$

$$k \geq k_{\min}$$

Moments

Distribution: $$P(k) = (\alpha - 1)k_{\min}^{\alpha-1}k^{-\alpha}$$

(central) Moments: $$\langle k^m \rangle = \int_{k_{\min}}^{\infty} k^m p(k)dk$$

Reminder:

$\langle k^1 \rangle$ Average
$\langle k^2 \rangle$ Variance
$\langle k^3 \rangle$ Skewness
…

Moments

Distribution: $P(k) = (\alpha - 1)k_{\min}^{\alpha-1}k^{-\alpha}$

(central) Moments:

$$\langle k^m \rangle = \int_{k_{\min}}^{\infty} k^m p(k)dk$$

$$\langle k^m \rangle = (\alpha - 1)k_{\min}^{\alpha-1}\int_{k_{\min}}^{\infty} k^{-\alpha+m}dk$$

$$\langle k^m \rangle = k_{\min}^{m}\left(\frac{\alpha - 1}{\alpha - 1 - m}\right)$$

Defined for $\alpha > m + 1$,
Otherwise diverge (+inf)

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$$

http://tuvalu.santafe.edu/~aaronc/courses/7000/csci7000-001_2011_L2.pdf

Moments

Distribution: $$P(k) = \frac{\alpha - 1}{k_{\min}} \left(\frac{k}{k_{\min}} \right)^{-\alpha}$$

(central) Moments: $$\langle k^m \rangle = k_{\min}^m \left(\frac{\alpha - 1}{\alpha - 1 - m} \right)$$

Defined for $\alpha > m + 1$,
Otherwise diverge (+inf)

=> Mean: $$\langle k \rangle = \frac{\alpha - 1}{\alpha - 2} k_{\min}$$

(But diverges for $\alpha \leq 2$)

=> Variance: $$\langle k \rangle = \frac{\alpha - 1}{\alpha - 3} k_{\min}^2$$

(But diverges for $\alpha \leq 3$)

Moments

What does divergence means in practice ?

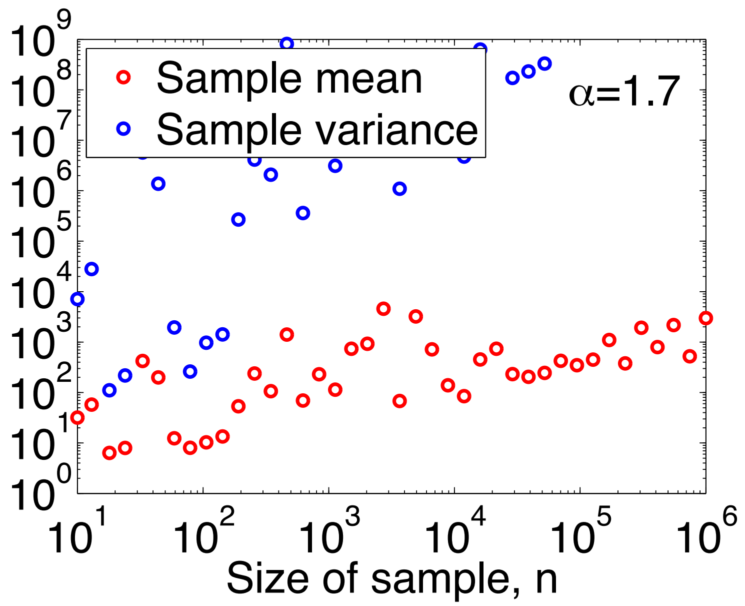
We can always *compute* the mean and variance, given samples of a distribution (e.g., an observe degree distribution)

=>The value computed depends on the **size** of the sample, it is not a characteristic of the distribution.

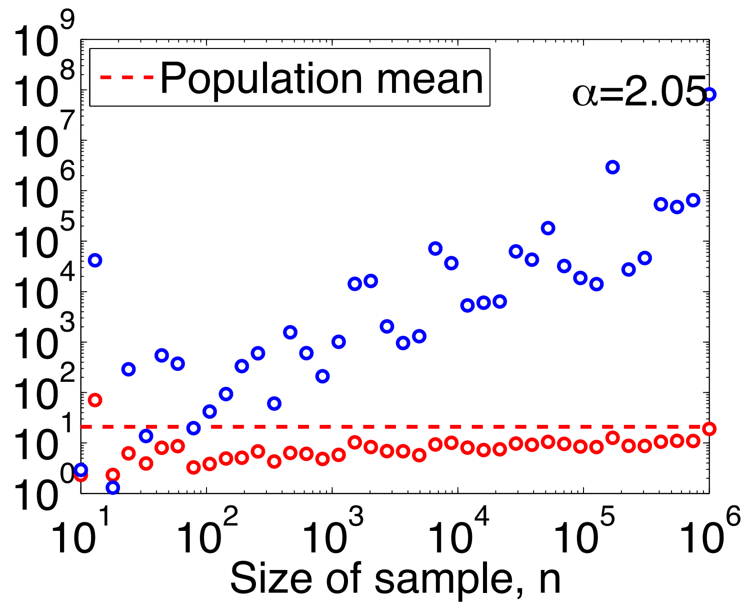
Moments are *dominated* by elements in the long tail. Some events are rare, but they have so large values, that if observed, they are strong enough to modify substantially the corresponding moment. And they appear frequently enough so that the mean will continue to shift when increasing the sample size

Moments

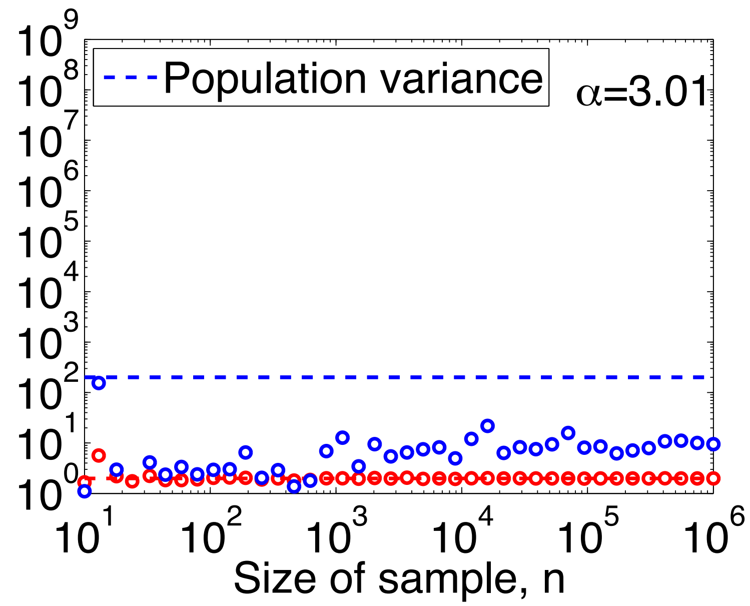
$\alpha < 2$
Mean diverge



$2 < \alpha < 3$
Mean well defined,
Variance diverge



$\alpha > 3$
Mean and variance defined



=> Even when well defined, **moments converge very slowly**

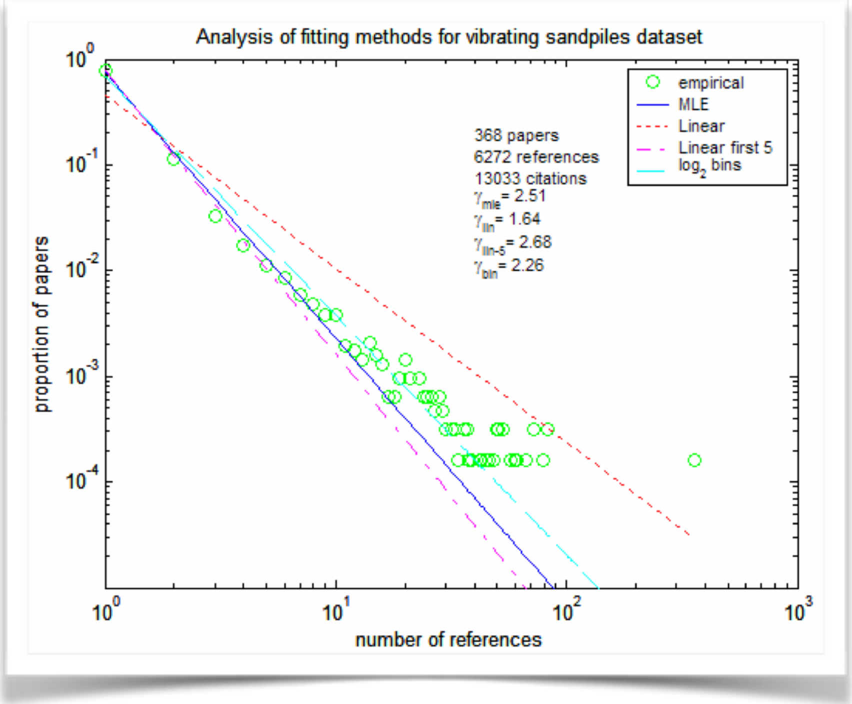
Scale-free networks

Computing the exponent of an observed network

Method 1: find the slope of the line of the log-log plot

Problem: most of data is on first value, so we *overfit* based on a
Few values in the long tail

More advanced method:
Maximum Likelihood Estimation (MLE)



[Fitting to the Power-Law Distribution, Goldstein et al.]
https://arxiv.org/vc/cond-mat/papers/0402/0402322v1.pdf

Scale-free networks

Exponent

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference
WWW	325 729	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, and Barabási 1999
WWW	4×10^7	7		2.38	2.1				Kumar *et al.*, 1999
WWW	2×10^8	7.5	4000	2.72	2.1	16	8.85	7.61	Broder *et al.*, 2000
WWW, site	260 000				1.94				Huberman and Adamic, 2000
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2	4	6.3	5.2	Faloutsos, 1999
Internet, router*	3888	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos, 1999
Internet, router*	150 000	2.66	60	2.4	2.4	11	12.8	7.47	Govindan, 2000
Movie actors*	212 250	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási and Albert, 1999
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2	4	2.12	1.95	Newman, 2001b
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.*, 2001
Co-authors, math.*	70 975	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.*, 2001
Sexual contacts*	2810			3.4	3.4				Liljeros *et al.*, 2001
Metabolic, *E. coli*	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.*, 2000
Protein, *S. cerev.**	1870	2.39		2.4	2.4				Jeong, Mason, *et al.*, 2001
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya and Solé, 2000
Silwood Park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya and Solé, 2000
Citation	783 339	8.57			3				Redner, 1998
Phone call	53×10^6	3.16		2.1	2.1				Aiello *et al.*, 2000
Words, co-occurrence*	460 902	70.13		2.7	2.7				Ferrer i Cancho and Solé, 2001
Words, synonyms*	22 311	13.48		2.8	2.8				Yook *et al.*, 2001b

Albert, R. et.al. Rev. Mod. Phy. (2002)

Exponents of real-world networks are usually between 2 and 3

Scale-free networks

Exponent

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}
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WWW, site	260 000				1.94
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2
Internet, router*	3888	2.57	30	2.48	2.48
Internet, router*	150 000	2.66	60	2.4	2.4
Movie actors*	212 250	28.78	900	2.3	2.3
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1
Co-authors, math.*	70 975	3.9	120	2.5	2.5
Sexual contacts*	2810			3.4	3.4
Metabolic, *E. coli*	778	7.4	110	2.2	2.2
Protein, *S. cerev.**	1870	2.39		2.4	2.4
Ythan estuary*	134	8.7	35	1.05	1.05
Silwood Park*	154	4.75	27	1.13	1.13
Citation	783 339	8.57			3
Phone call	53×10^6	3.16		2.1	2.1
Words, co-occurrence*	460 902	70.13		2.7	2.7
Words, synonyms*	22 311	13.48		2.8	2.8

Albert, R. et.al. Rev. Mod. Phy. (2002)

- Average values are not reliable since the convergence is very slow
- Furthermore, average values are meaningless since the fluctuations are infinitely large (diverging variance)

Scale-free networks

Why do most of the real networks have degree exponent between 2 and 3?

- If the exponent is smaller than 2, the distribution is so skewed that we expect to find nodes with a degree *larger* than the size of the network => not possible in finite networks

Scale-free networks

Why do most of the real networks have degree exponent between 2 and 3?

- To detect a scale-free network its degree distribution needs to span through several (at least 2-3) orders of magnitude ⇒ $K_{max} \sim 10^3$
- If the exponent is large (>3), large degrees become so rare that the size of the sample (i.e., size of observed network) must be enormous to indeed observe such an edge
- Example: let's choose $\gamma=5$, $K_{min}=1$ and $K_{max} \sim 10^3$

$$K_{\max} = K_{\min} N^{\frac{1}{\gamma-1}}$$

$$N = \left(\frac{K_{\max}}{K_{\min}}\right)^{\gamma-1} \approx 10^{12}$$

We need to observe 10^{12} nodes to observe a node of degree 1000 for exponent=5

=> Forget about (single planet) social networks…

Scale-free networks - distances

$$
\langle l \rangle \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}
$$

Ultra Small World ($2 < \gamma < 3$)

Small World ($\gamma > 3$)

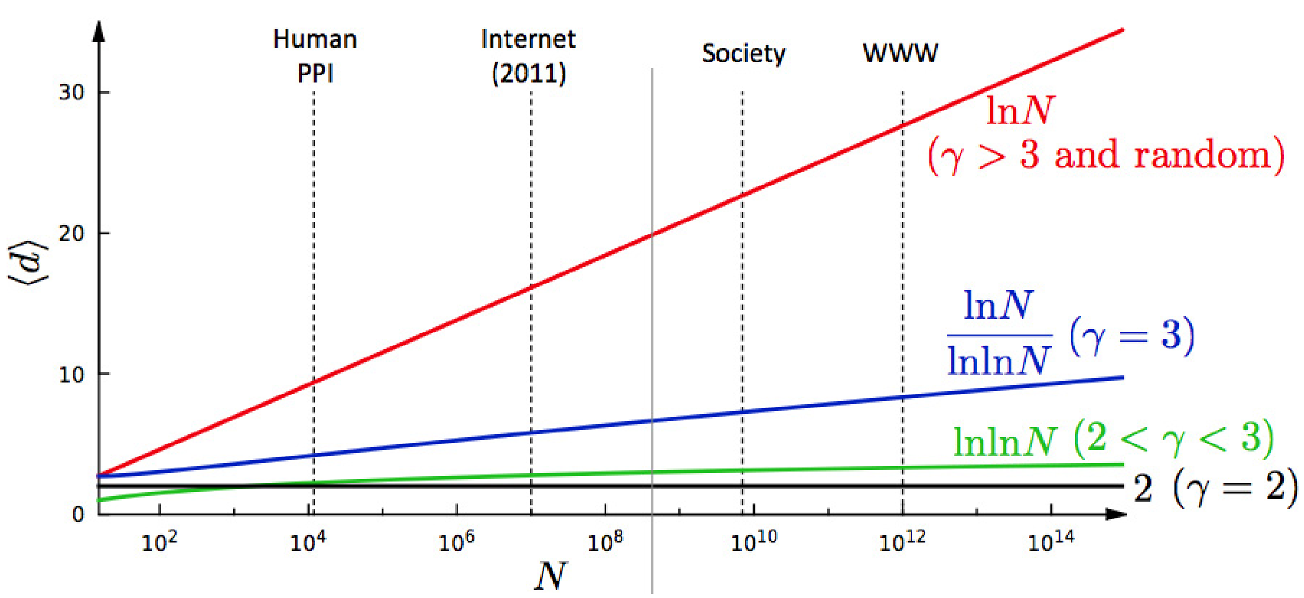
$\gamma = 2$: Size of the biggest hub is of order O(N). Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.

$2 < \gamma < 3$: The average path length increases slower than logarithmically. In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

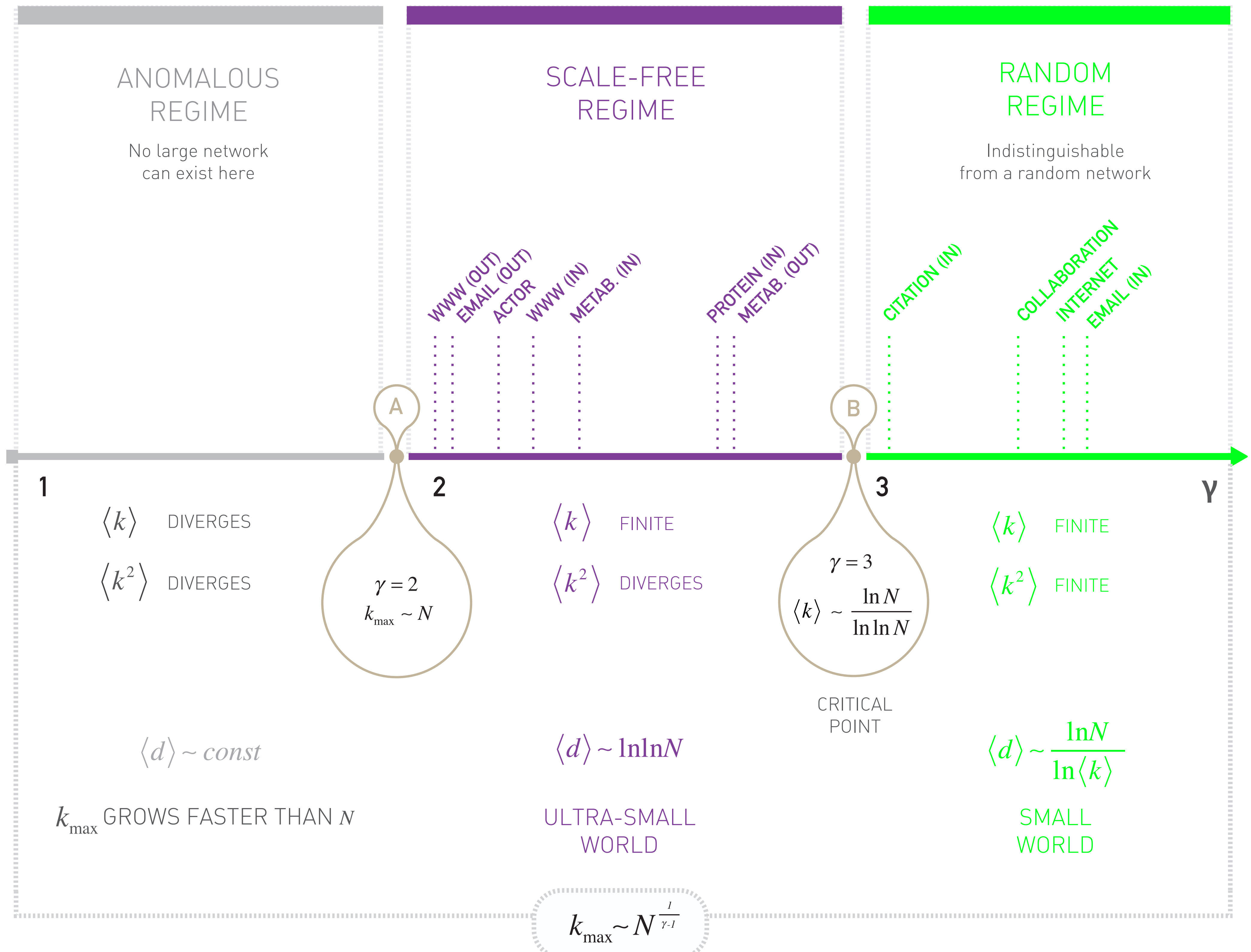
$\gamma = 3$: Some key models produce γ=3, so the result is of particular importance for them. This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

$\gamma > 3$: The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

Cohen, Havlin Phys. Rev. Lett. 90, 58701(2003); Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks, Eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) Chap. 4; Confirmed also by: Dorogovtsev et al (2002), Chung and Lu (2002); (Bollobas, Riordan, 2002; Bollobas, 1985; Newman, 2001



Scale-free networks - summary



Slide from CCNR Complex Networks Course
A. L. Barabási 2014

Scale-free networks

- Are real networks really Scale Free ?
- In most real networks, the scale free stands only for *a range* of degrees, i.e., between a minimum degree and maximum degree different than those observed (cut-offs)
- Some other distributions, in particular log-normal distributions, might “look like” power-law

Albert-László Barabási

Aaron Clauset

Emergence of scaling in random networks (1999)

Scale-free networks are rare (2018)

Love is All You Need - Clauset's fruitless search for scale-free networks (2018)

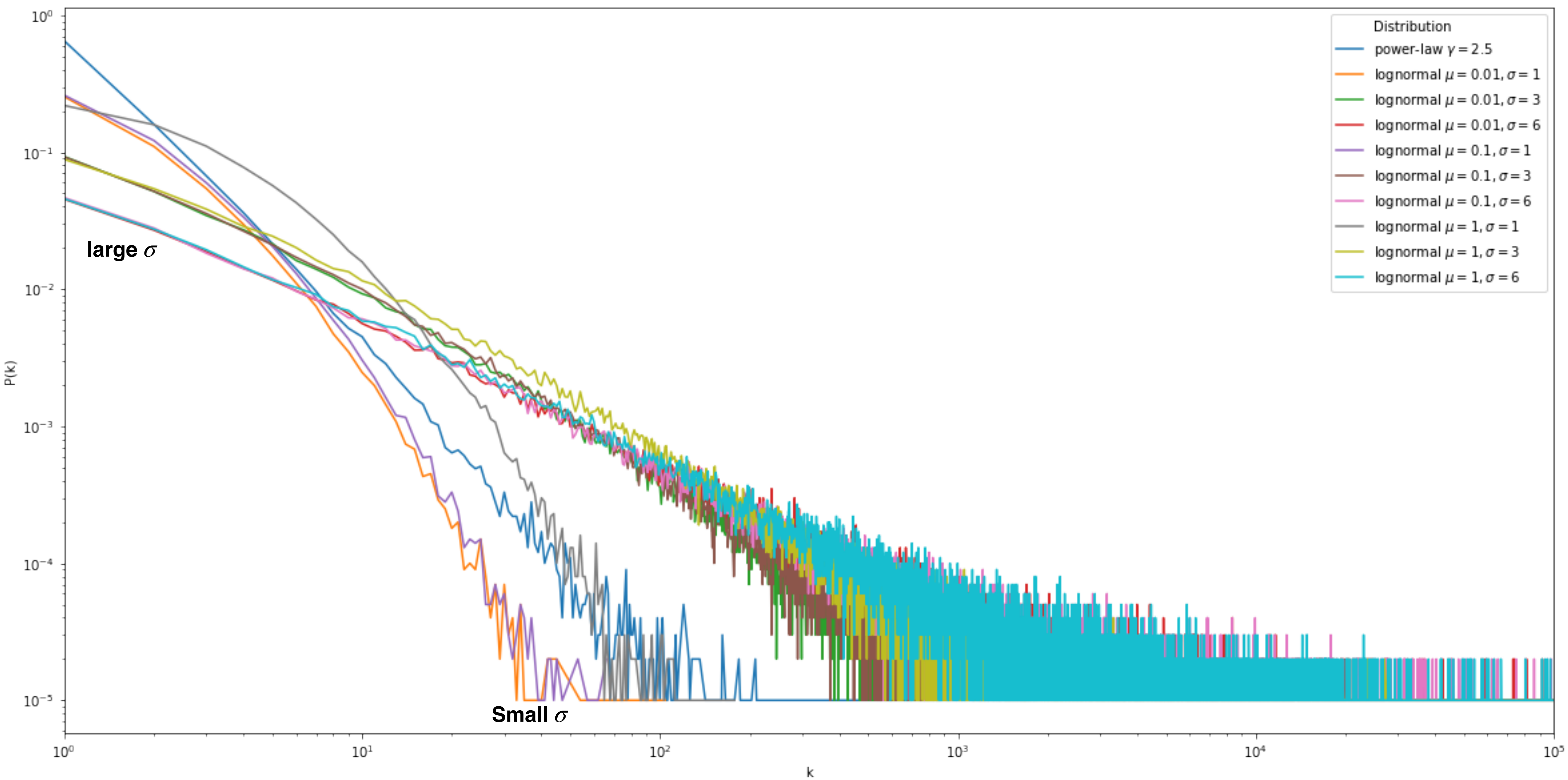
Petter Holme

Rare and everywhere: Perspectives on scale-free networks (2019)

Scale-free networks

Comparing a log-normal distribution and a power law

$$\frac{1}{k\sigma\sqrt{2\pi}}\ \exp\left(-\frac{(\ln k-\mu)^2}{2\sigma^2}\right) \qquad k^{-\alpha}$$



Log-normal distribution = Probability distribution of a random variable whose logarithm is normally distributed

μ, σ Mean, std of the log of the variable

Scale-free networks

Albert-László Barabási
@barabasi

@aaronclauset Every 5 years someone is shocked to re-discover that a pure power law does not fit many networks. True: Real networks have predictable deviations. Hence forcing a pure power law on these is like...fitting a sphere to the cow. Sooner or later the hoof will stick out.

4.13.4 Systematic Fitting Issues

The procedure described above may offer the impression that determining the degree exponent is a cumbersome but straightforward process. In reality, these fitting methods have some well-known limitations: **Network Science, Chapter 4, pg 159**

1. A pure power law is an idealized distribution that emerges in its form (4.1) only in simple models (**Chapter 5**). In reality, a whole range of processes contribute to the topology of real networks, affecting the precise shape of the degree distribution. These processes will be discussed in **Chapter 6**. If p_k does not follow a pure power law, the methods described above, designed to fit a power law to the data, will inevitably fail to detect statistical significance. While this finding can mean that the network is not scale-free, it most often means that we have not yet gained a proper understanding of the precise form of the degree distribution. Hence we are fitting the wrong functional form of p_k to the dataset.

Albert-László Barabási @barabasi · Jan 15, 2018
Replying to @barabasi
Chapter 6 in Network Science networksciencebook.com/chapter/6 discusses what you should be fitting to the degree distribution of *real* scale-free networks. You are right: Pure power laws are predictably rare. Scale-free networks are not.
1 21 45

Aaron Clauset @aaronclauset · Jan 15, 2018
Replying to @barabasi
Yes, science is hard and real data often messy. But it is worrying how criticisms of harsh statistical evaluations can be interpreted as a belief that "disagreement with data" (as Feynman would put it) should not be held against a favored theory or model.
3 5 18

Albert-László Barabási @barabasi · Jan 15, 2018
We are on the same page. The question is, what you test and what you conclude. There are multiple processes that contribute to the degree distribution that modify the power law. Hence testing for power laws only you are ignoring them all, leading to misleading takeway message.
2 4 10

Aaron Clauset @aaronclauset · Jan 15, 2018
Perhaps. I feel good about the accuracy of our conclusions: we used rigorous statistical methods, tested 5 distributions, considered 5 levels of evidence, across nearly 1000 network datasets. The goal was to be thorough and to treat the SF hypothesis as falsifiable.
1 3 14

Albert-László Barabási @barabasi · Jan 15, 2018
The effort is amazing. The conclusions are less so. The feather falls slower than the rock, yet gravitation is not wrong. We add friction. You need to fit for each system the Pk that is right for it. That is hard, I know. Otherwise you ignore 20 year of work by hundreds.
2 4 6

Aaron Clauset @aaronclauset · Jan 15, 2018
It seems easy to get confused here: an empirical power-law degree distribution is evidence for SF structure, but no deviation from the power law can be evidence against SF structure? It is reasonable to believe a fundamental phenomena would require less customized detective work.

Scale-free networks

Aaron Clauset

-Rigorous **statistical tests** show that observed degree distributions are not compatible with a power law distribution (high p-values)

-Compared with different distributions, in particular log-normal, most degree distributions are more likely to be generated by something else than power laws

Albert-László Barabási

-Networks are real objects, not mathematical abstraction, therefore they are sensible to **noise** (real life limits…)

-Power law is a **good, simple model** of degree distributions of a class of networks

-20 years of **fruitful research** based on this model

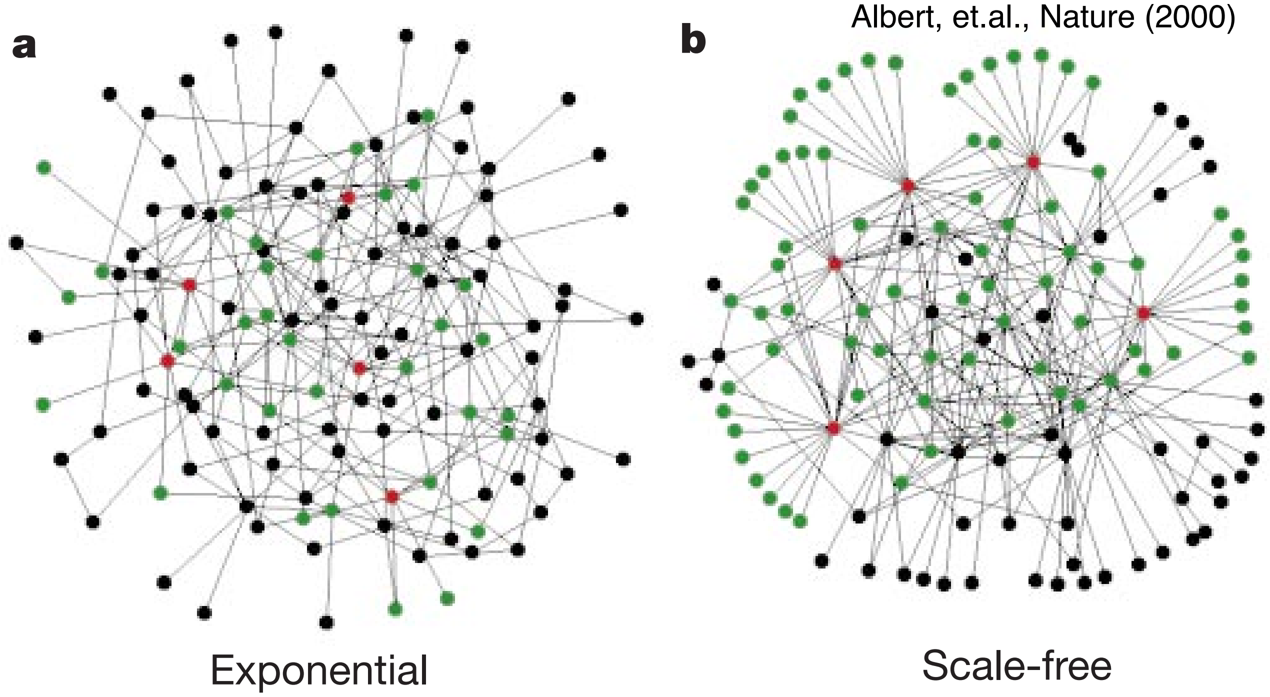
Scale-free networks
Robustness against
failures and attacks

Scale-free networks - role of hubs

Network robustness and attack tolerance

- How network topology is resistant against failure and targeted attacks

Albert, et.al., Nature (2000)



(a) Poisson random graph

(b) Scale-free network

Both networks have the same parameters:

- N=130
- <k>=3.3

Numerical experiment:

1. Take a connected network
2. Remove nodes one at a time
3. Observe the size of the LCC

Node removal strategies:

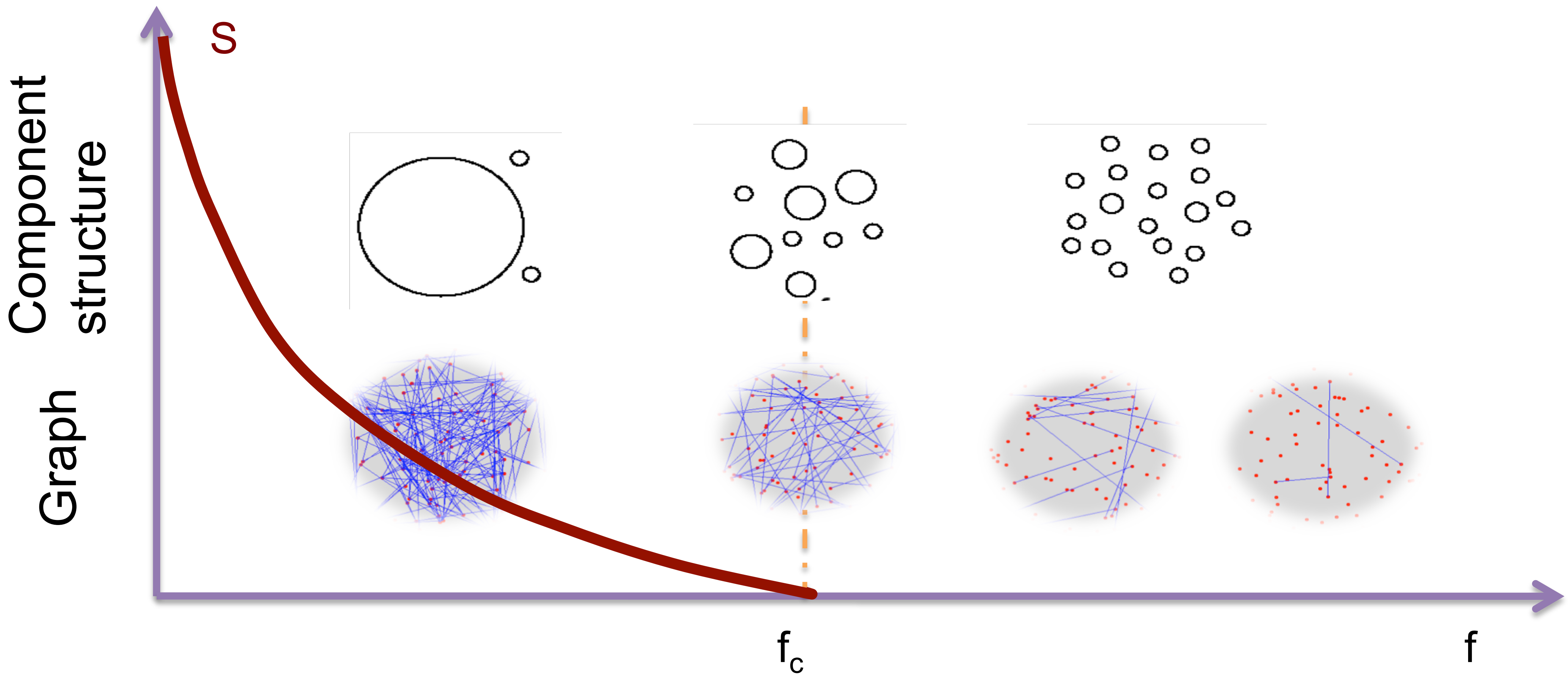
Remove nodes randomly (“failures”)

Example: Random failure of routers on the internet

Removal of nodes from networks

Inverse percolation problem

f= fraction of removed nodes



(Inverse Percolation phase transition)

Malloy-Reed criteria for giant components

A giant cluster exists if each node is connected to at least two other nodes.

The average degree of a node i linked to the GC, must be 2.

Can be shown to correspond to the following relation:

$$\boxed{\kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} = 2}$$

κ>2: Giant component exist

κ<2: Many disconnected cluster

Malloy, Reed, Random Structures and Algorithms (1995); Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

Breakdown threshold for ER networks

Random node removal changes

- The degree of individual nodes *[k -> k' ≤ k]* decrease by losing links via node removal
- A node with degree k becomes a node with degree k' with probability:

$$\binom{k}{k'} f^{k-k'}(1-f)^{k'} \qquad \text{where} \qquad k' \leq k$$

Remove k-k' links, each with probability f

Leave k' links untouched, each with probability 1-f

- the degree distribution *[P(k) -> P'(k')]* after random removal of f fraction of nodes

$$P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'}(1-f)^{k'}$$

Malloy, Reed, Random Structures and Algorithms (1995); Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

Breakdown threshold for arbitrary $P(k)$

Malloy, Reed, Random Structures and Algorithms (1995); Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

$$P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^{k'}$$

$$\boxed{\langle k' \rangle_f = (1-f)\langle k \rangle} \qquad \boxed{\langle k'^2 \rangle_f = (1-f)^2 \langle k^2 \rangle + f(1-f)\langle k \rangle}$$

We know:

$$\langle k' \rangle_f = (1-f)\langle k \rangle$$

$$\langle k'^2 \rangle_f = (1-f)^2 \langle k^2 \rangle + f(1-f)\langle k \rangle$$

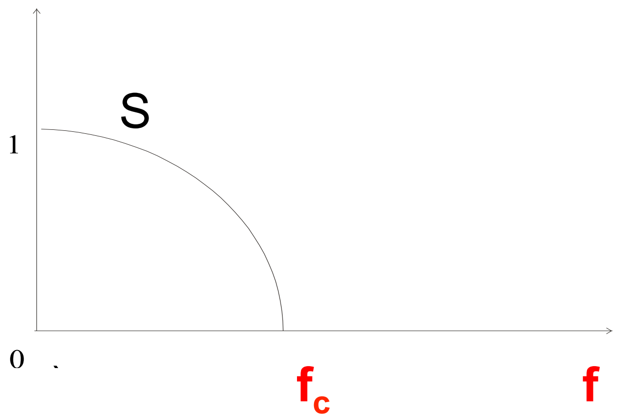
$$\kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} = 2$$

κ>2: Giant component exist

κ<2: Many disconnected cluster

Breakdown threshold:

$$f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}$$



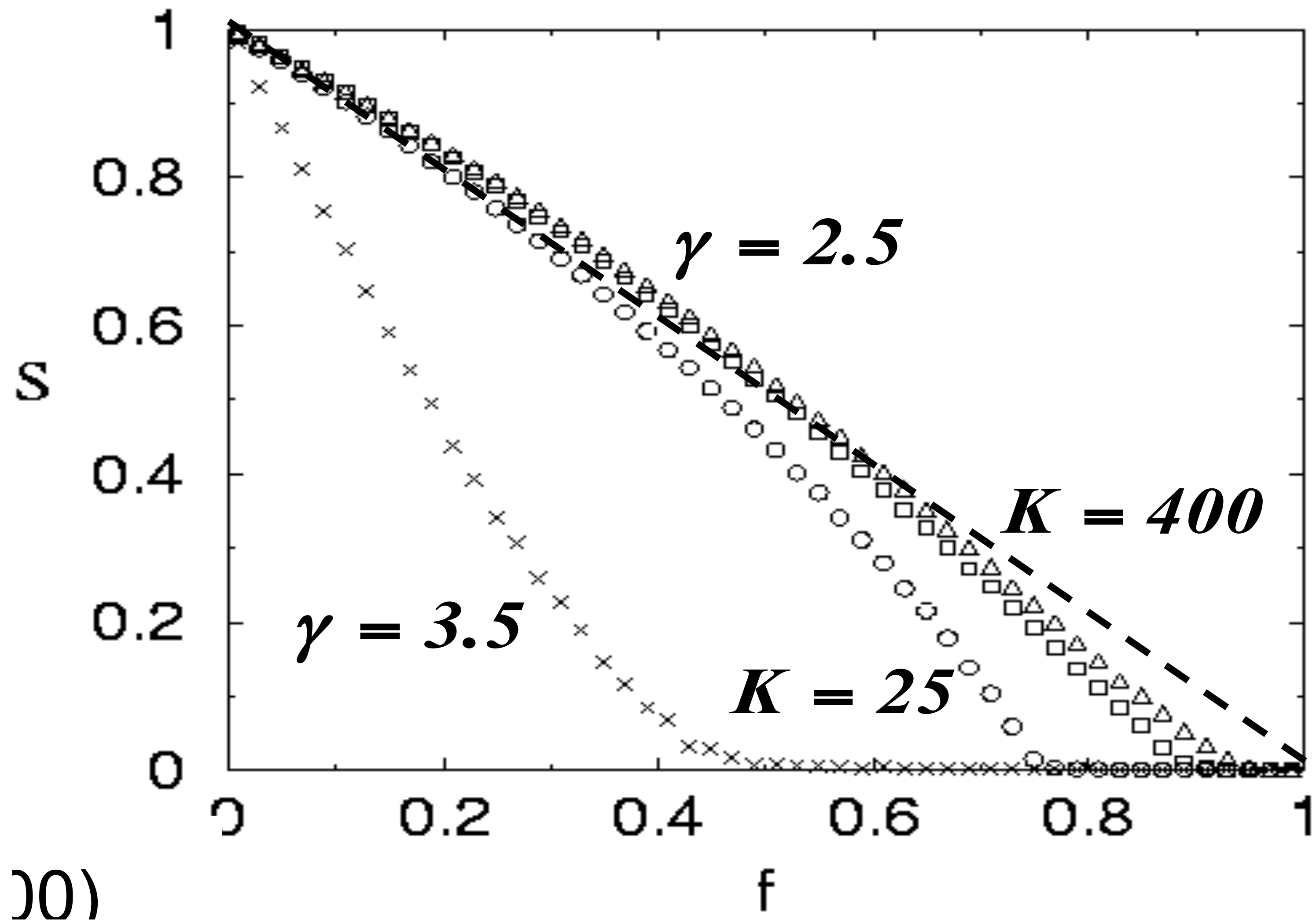
$$\kappa \cong 1 - CN^{-\frac{3-\gamma}{\gamma-1}}$$

Robustness of scale-free networks

Scale-free random graph with

$P(k) = Ak^{-\gamma}$ with $k=m,\ldots K$

$$\kappa \cong 1 - CN^{-\frac{3-\gamma}{\gamma-1}}$$



Cohen et al., Phys. Rev. Lett. 85, 4626 (2000)

Internet:

Router level map, $N=228{,}263$; $\gamma=2.1\pm0.1$; $\kappa=28$ -> $f_c=0.962$

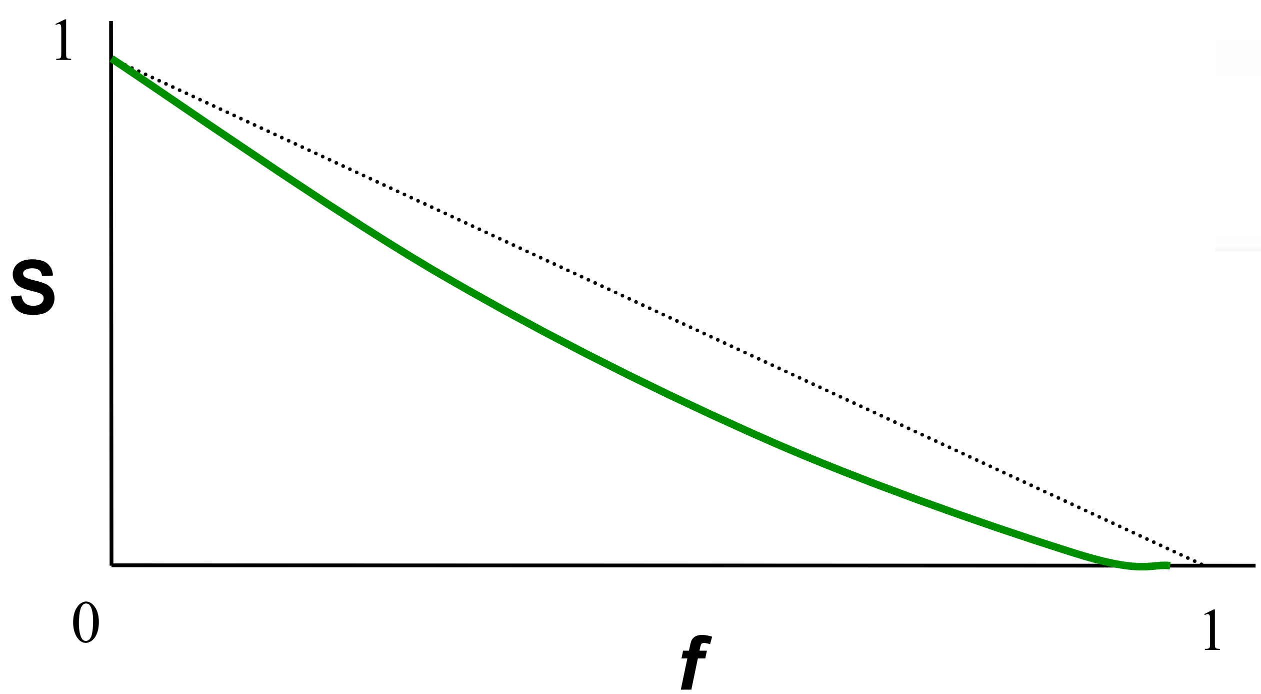
AS level map, $N= 11{,}164$; $\gamma=2.1\pm0.1$; $\kappa=264$ -> $f_c=0.996$

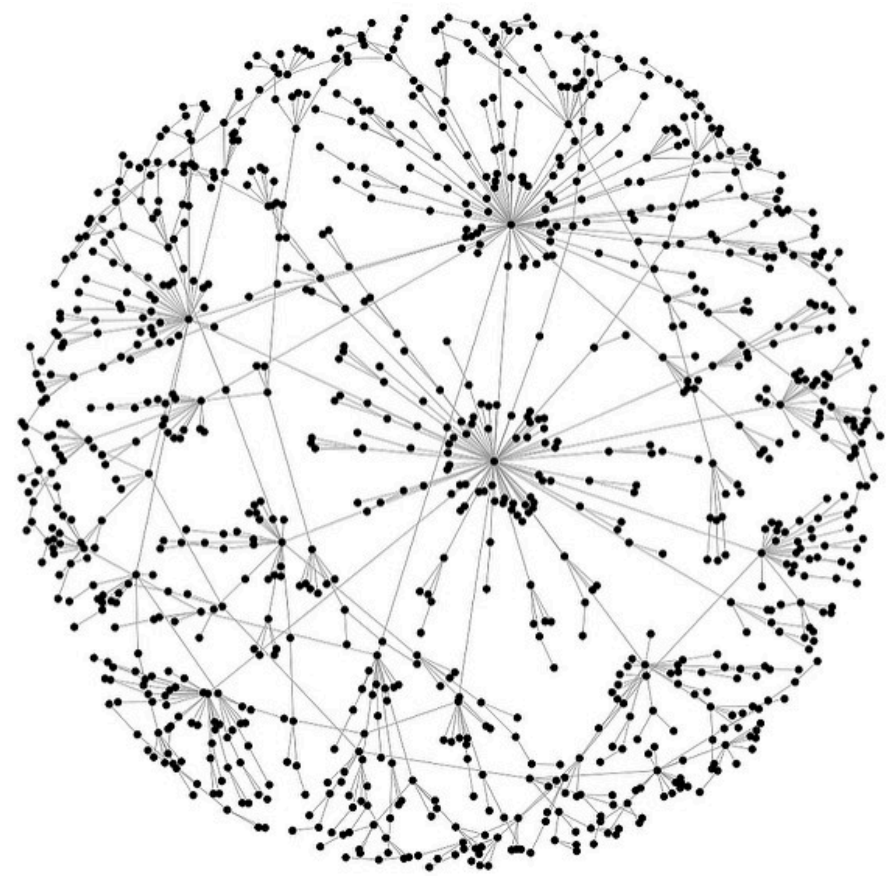
Robustness of scale-free networks

Scale-free networks do not appear to break apart under random failures.

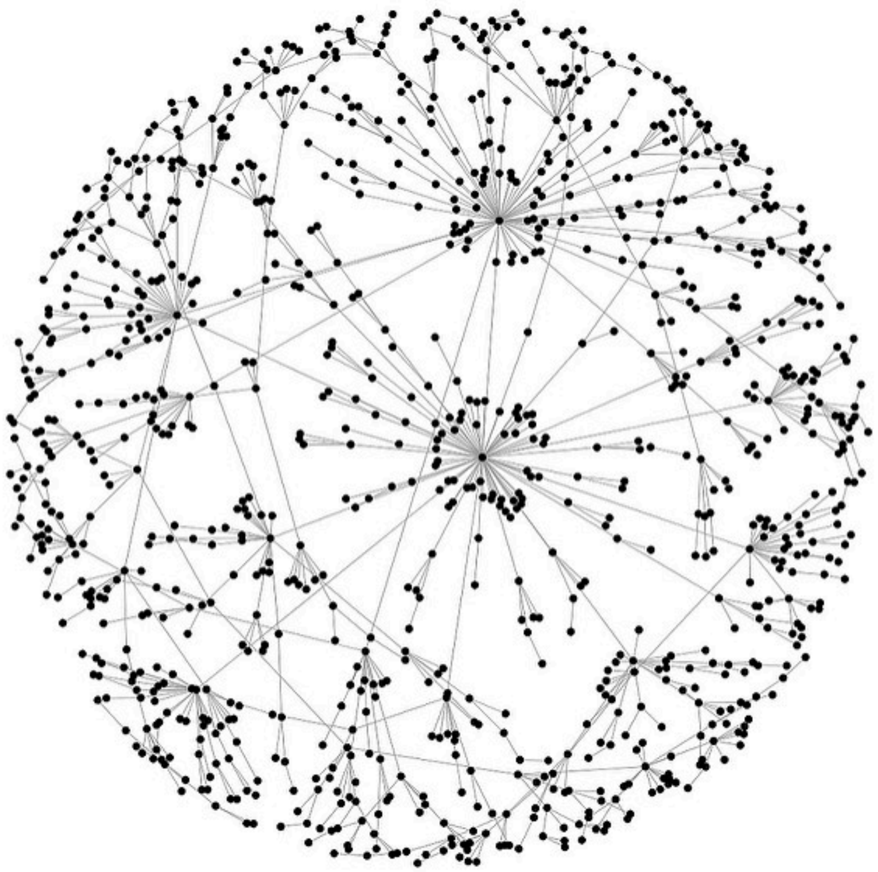
Reason: the likelihood of removing a hub is small.

Albert, Jeong, Barabási, *Nature* **406** 378 (2000)





Achilles' Heel of scale-free networks



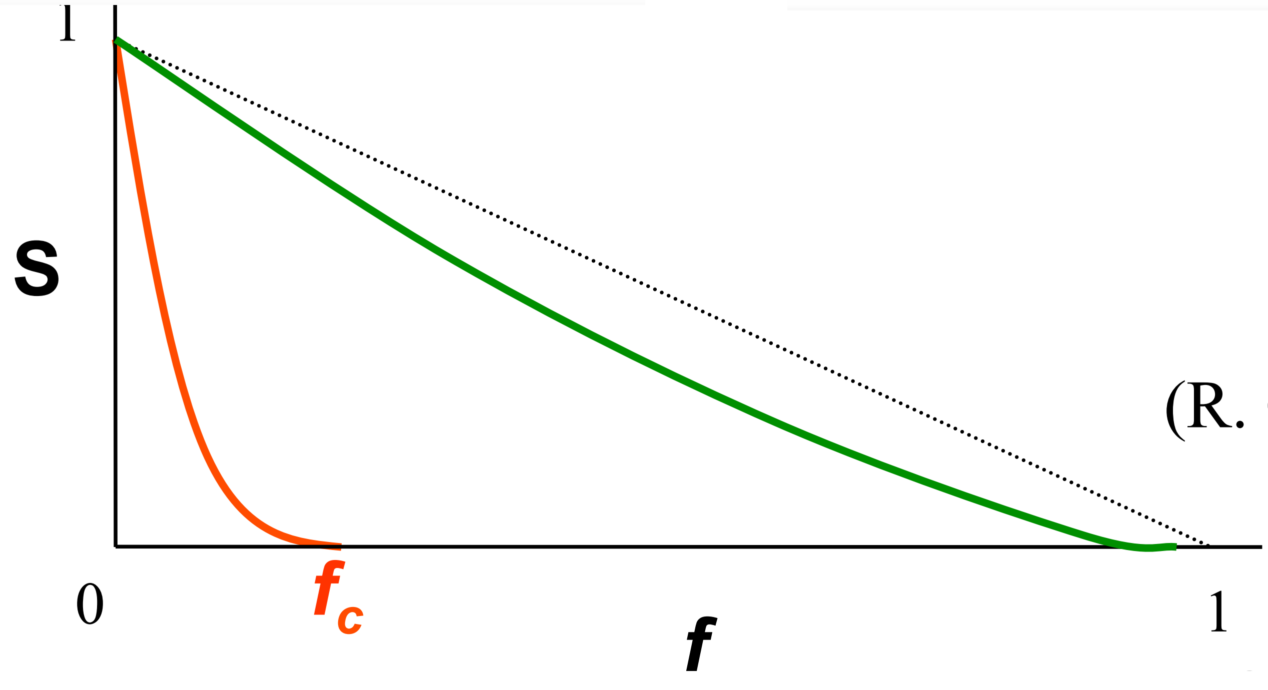
The robustness of scale free networks is due to the hubs, which are difficult to hit by chance.

Node removal strategies:

Remove nodes in descending order of their degrees, i.e. hubs first ("attacks")

Examples: Terrorist attacks, efficient vaccination in epidemics

Attacks



Failures

$\gamma \leq 3 : f_c = 1$

(R. Cohen et al PRL, 2000)

Attack threshold for scale-free networks

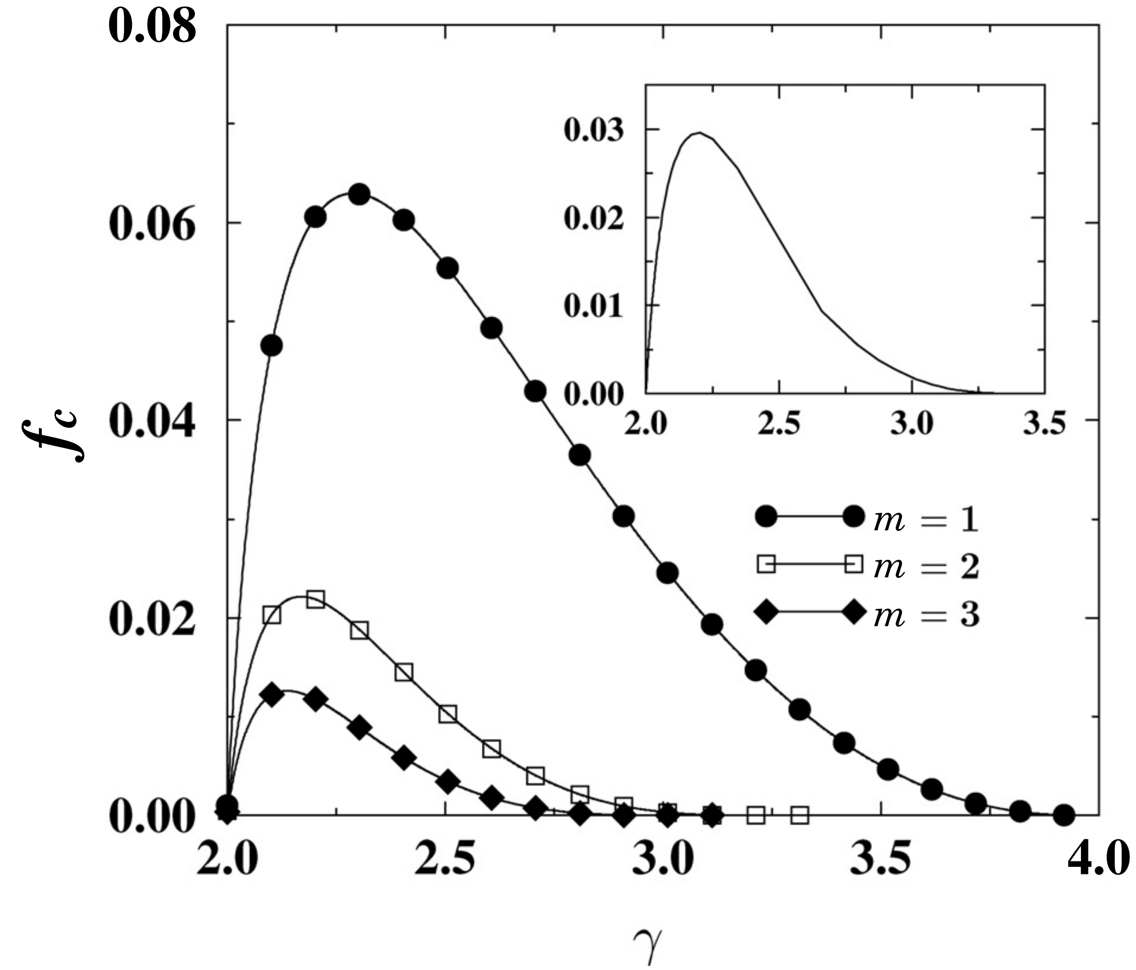
Attack problem: we remove a fraction f of the hubs.

At what threshold f_c will the network fall apart (no giant component)?

Critical threshold for scale-free networks:

$$f_c^{\frac{2-\gamma}{1-\gamma}} = 2 + \frac{2-\gamma}{3-\gamma} K_{min} \left(f_c^{\frac{3-\gamma}{1-\gamma}} - 1 \right)$$

- f_c depends on γ; it reaches its max for $\gamma<3$
- f_c depends on K_{min} (m in the figure)
- **Most important**: f_c is tiny. Its maximum reaches only 6%, i.e. **the removal of 6% of nodes can destroy the network in an attack mode.**
- **Internet**: $\gamma=2.1$, so 4.7% is the threshold.

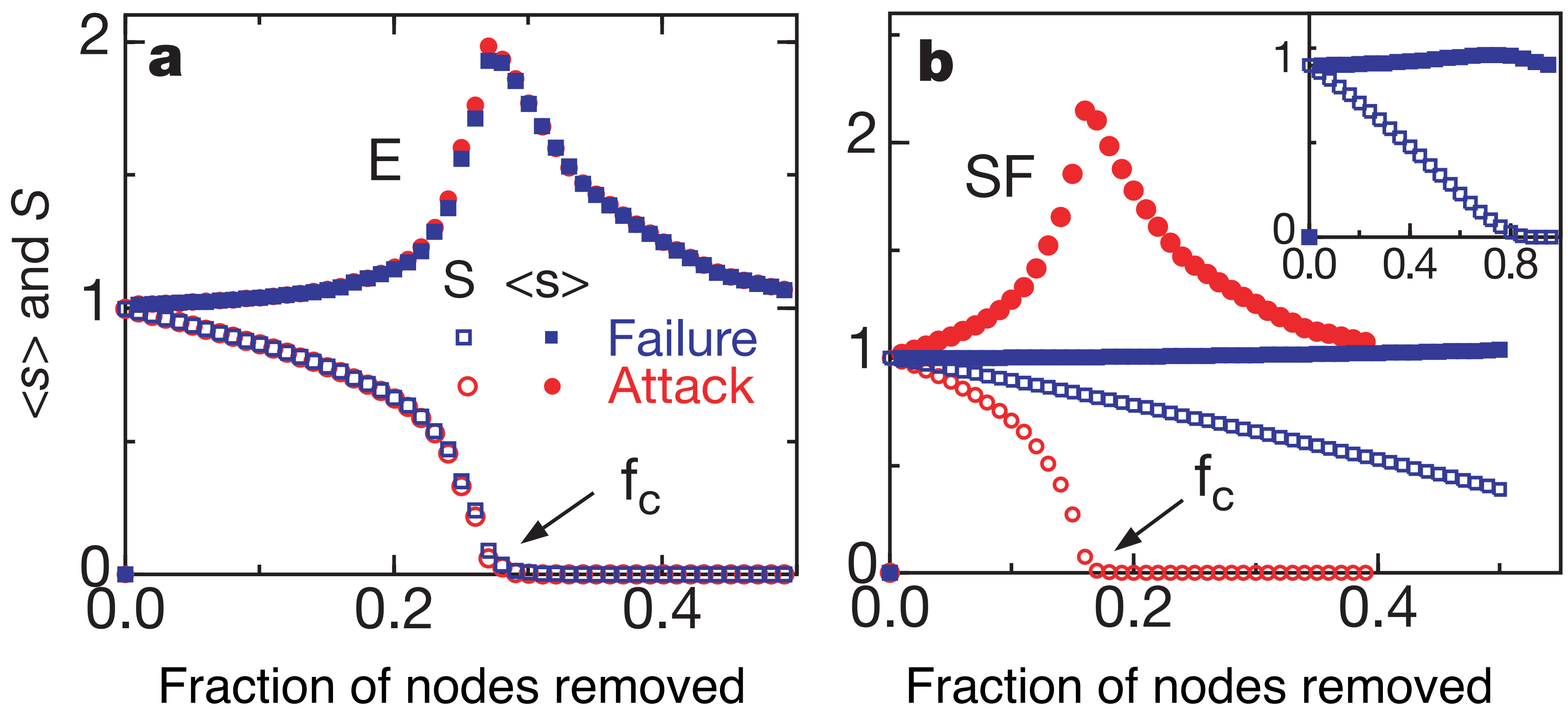


Pastor-Satorras, Vespignani, Evolution and Structure of the Internet (Cambridge University Press, 2004)

Scale-free networks - role of hubs

Network robustness and attack tolerance

Albert, et.al., Nature (2000)



- S: relative size of the LCC
- $<s>$: average size of components other than LCC

Poisson random graph

- Both removal methods give the same result
- The network falls apart after a finite fraction of nodes are removed ($S \to 0$)

Scale-free network

- Robust against random removal (blue)
- Vulnerable against targeted attacks

Consequences:

- Internet is still working even several servers are out of service
- Random vaccination is not effective in case of epidemic spreading

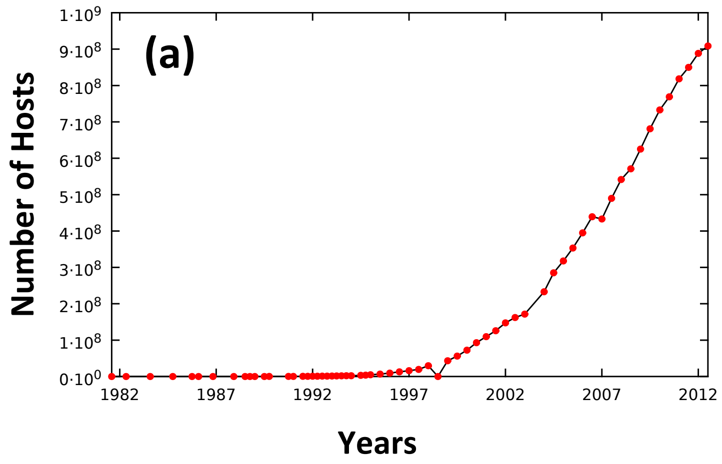
The Barabási-Albert model

of scale-free networks

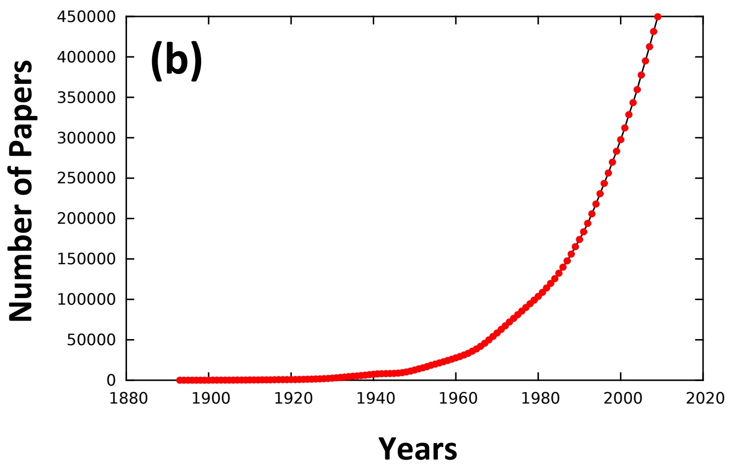
Emergence of hubs

What did we miss with the earlier network models?

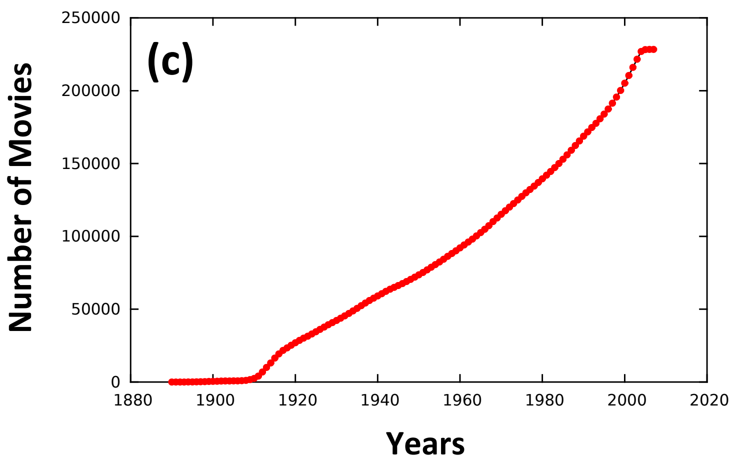
1. Networks are evolving
    - Networks are not static but growing in time as new nodes are entering the system



2. Preferential attachement
    - Nodes are not connected randomly but tends to link to more attractive nodes



- Pólya urn model (1923)
- Yule process (1925)
- Zipf's law (1941)
- Cumulative advantage (1968)
- Preferential attachement (1999)
- Pareto's law - 80/20 rule
- The rich get richer phenomena
- etc.



AL Barabási, Network Science Book (2013)

The Barabási-Albert model

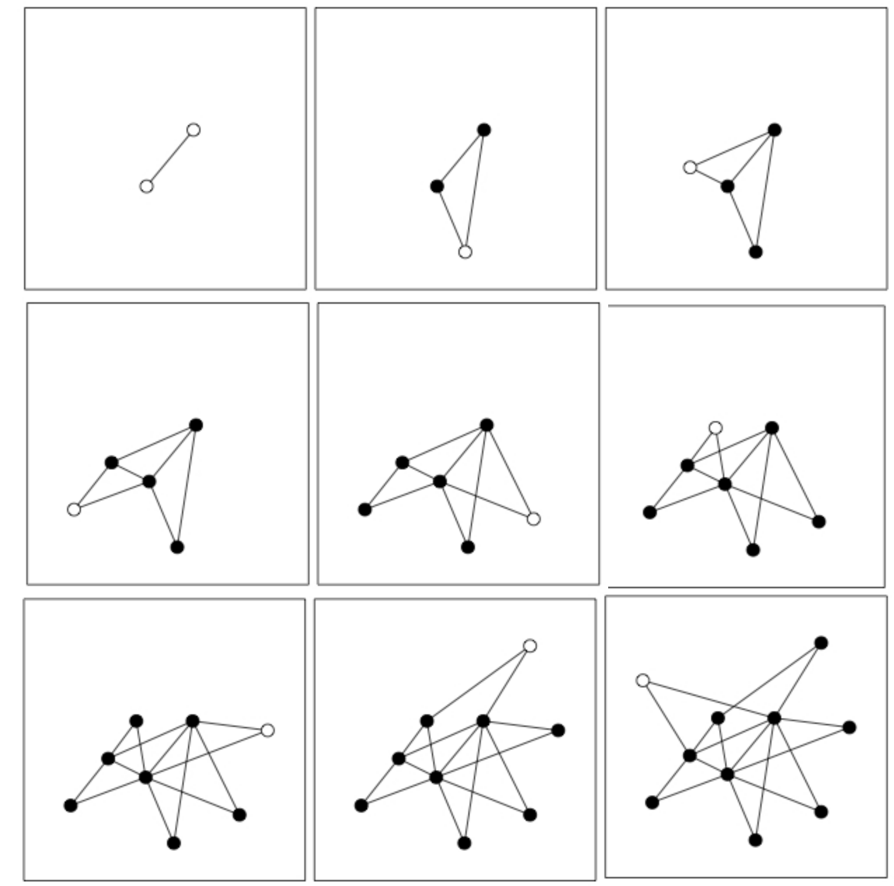
1. Start with m_0 connected nodes

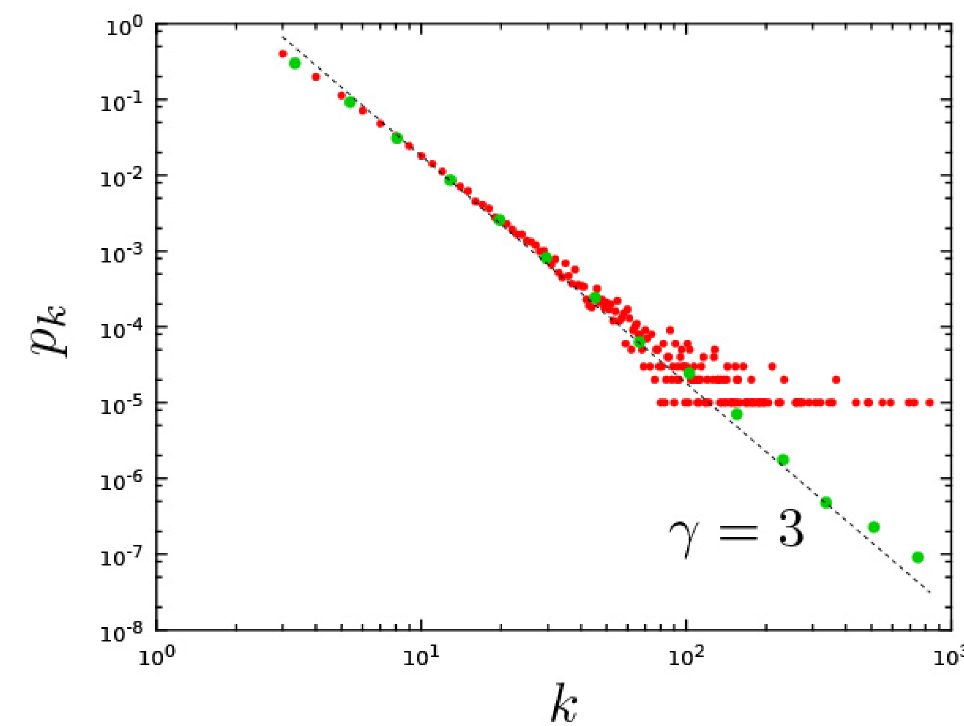
2. At each timestep we add a new node with $m\ (\leq m_0)$ links that connect the new node to m nodes already in the network.

3. The probability $\pi(k)$ that one of the links of the new node connects to node i depends on the degree k_i of node i as

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

- The emerging network will be scale-free with degree exponent $\gamma=3$ independently from the choice of m_0 and m

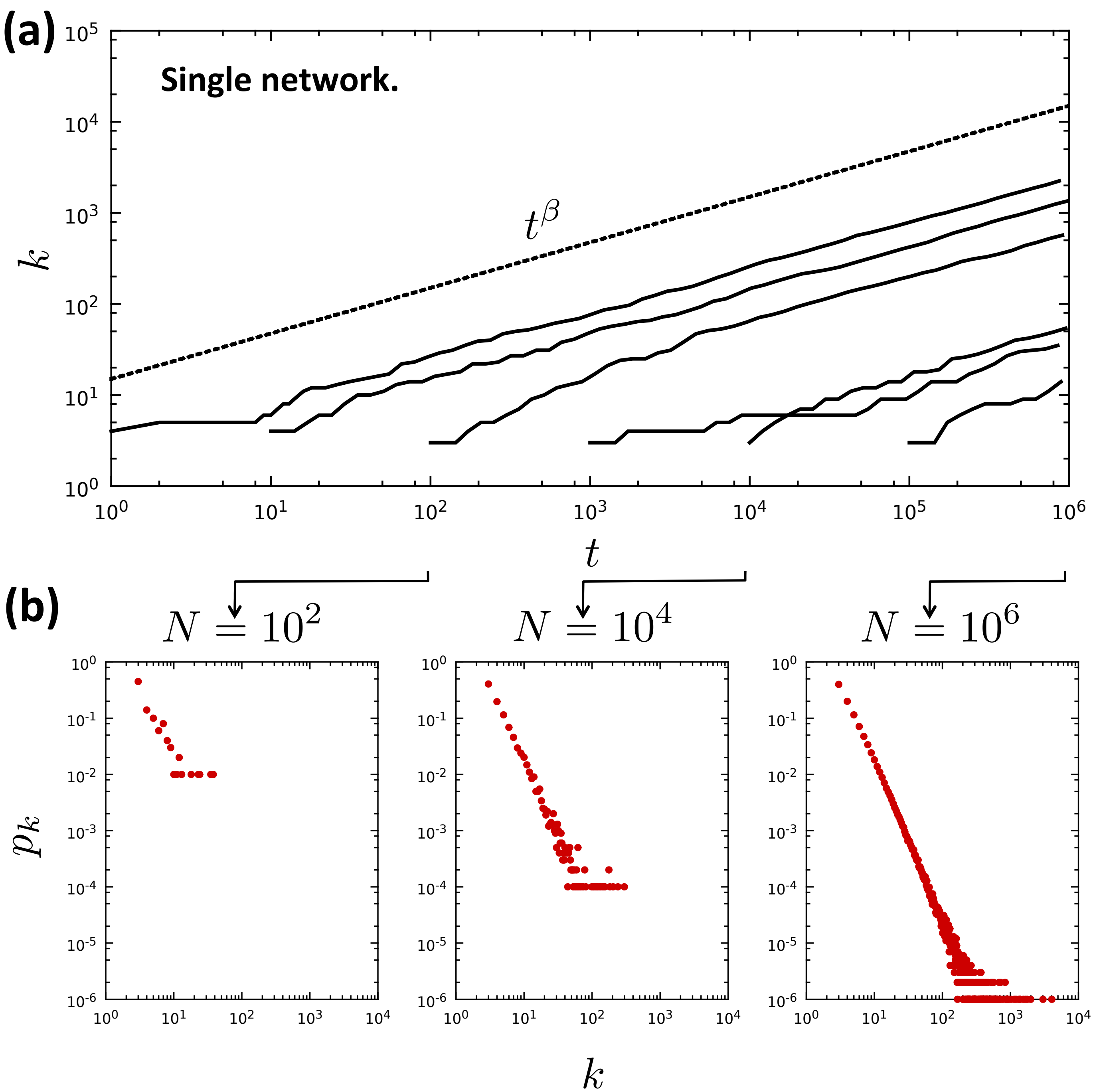




AL Barabási, Network Science Book (2013)

The BA model - emergence of hubs

solution by A.-L.Barabási, R. Albert and H. Jeong, Physica A 272, 173 (1999)



- The degree of each node increases as a power-law with exponent $\beta=1/2$
- Earlier a node was added larger its degree due to its earlier arrival and not because it grows feaster

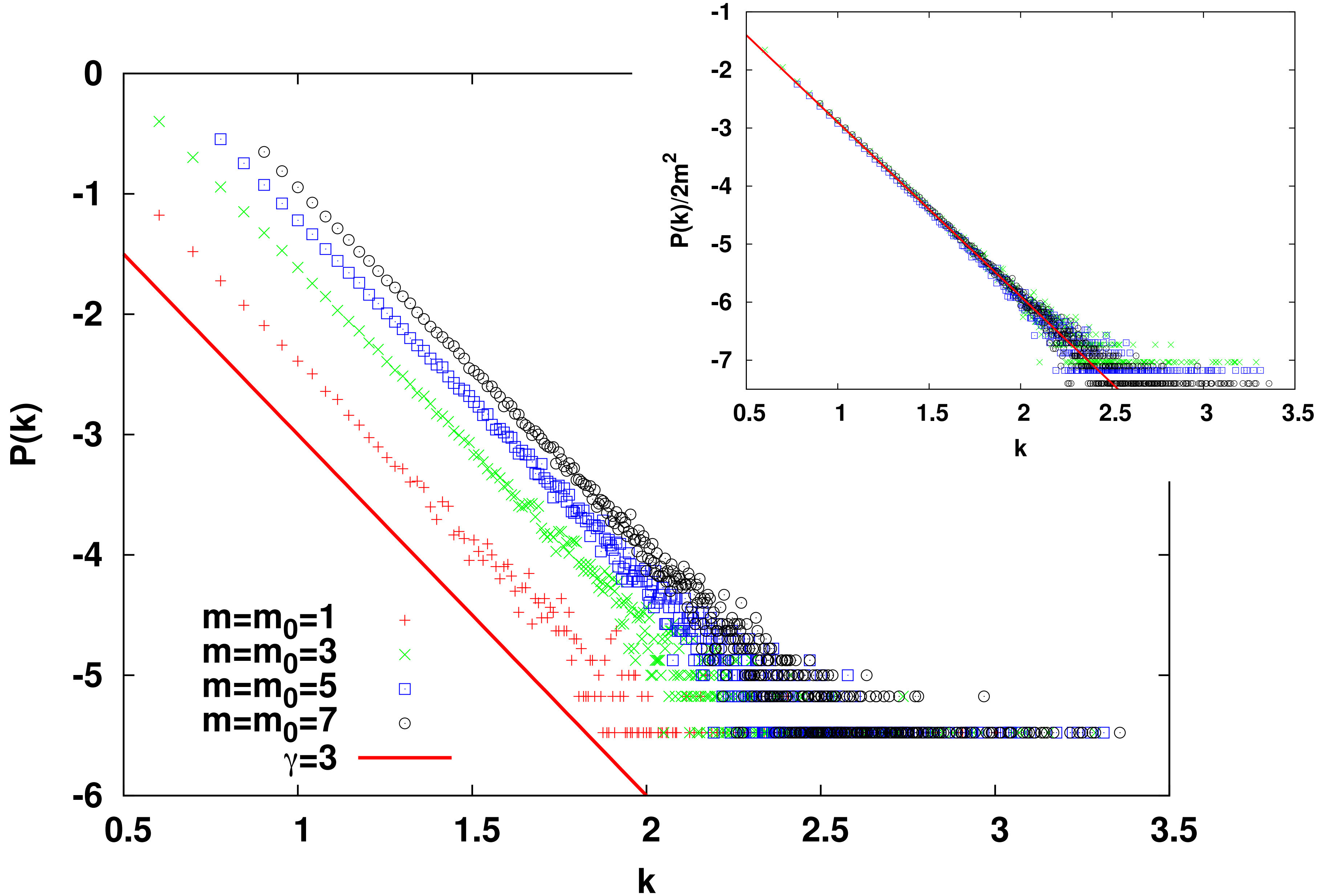
Rich-get-richer mechanism

AL Barabási, Network Science Book (2013)

The BA model - degree distribution

solution by A.-L.Barabási, R. Albert and H. Jeong, Physica A 272, 173 (1999)

- The degree exponent is independent of m
- The degree exponent is stationary in time and the degree distribution is time independent
- The exponent is compatible to the exponents of real networks



The BA model - path length

$$\langle l \rangle \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln\ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln\ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Ultra Small World

Small World

$\gamma = 2$: Size of the biggest hub is of order O(N). Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.

$2 < \gamma < 3$: The average path length increases slower than logarithmically. In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

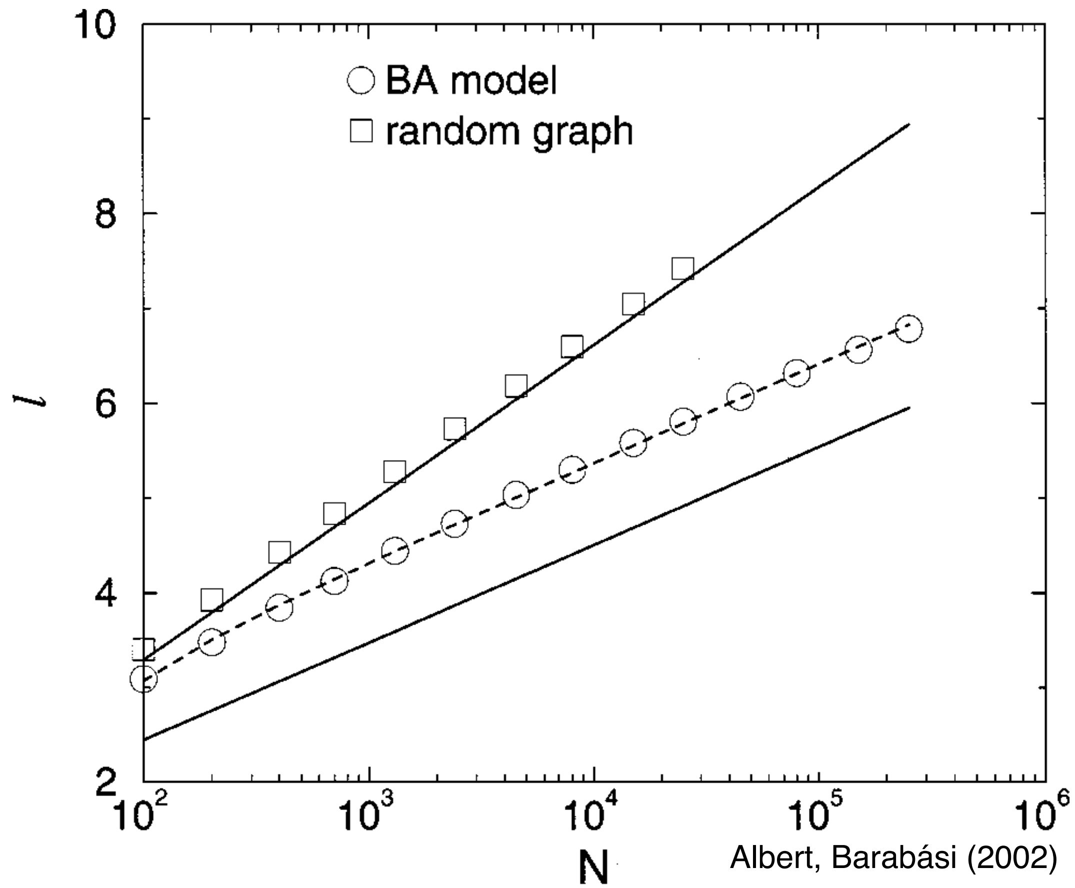
$\gamma = 3$: Some key models produce γ=3, so the result is of particular importance for them. This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

$\gamma > 3$: The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

$$\langle l \rangle = \frac{\ln N}{\ln\ln N}$$

Ultra Small World network

Bollobás, Riordan (2001)



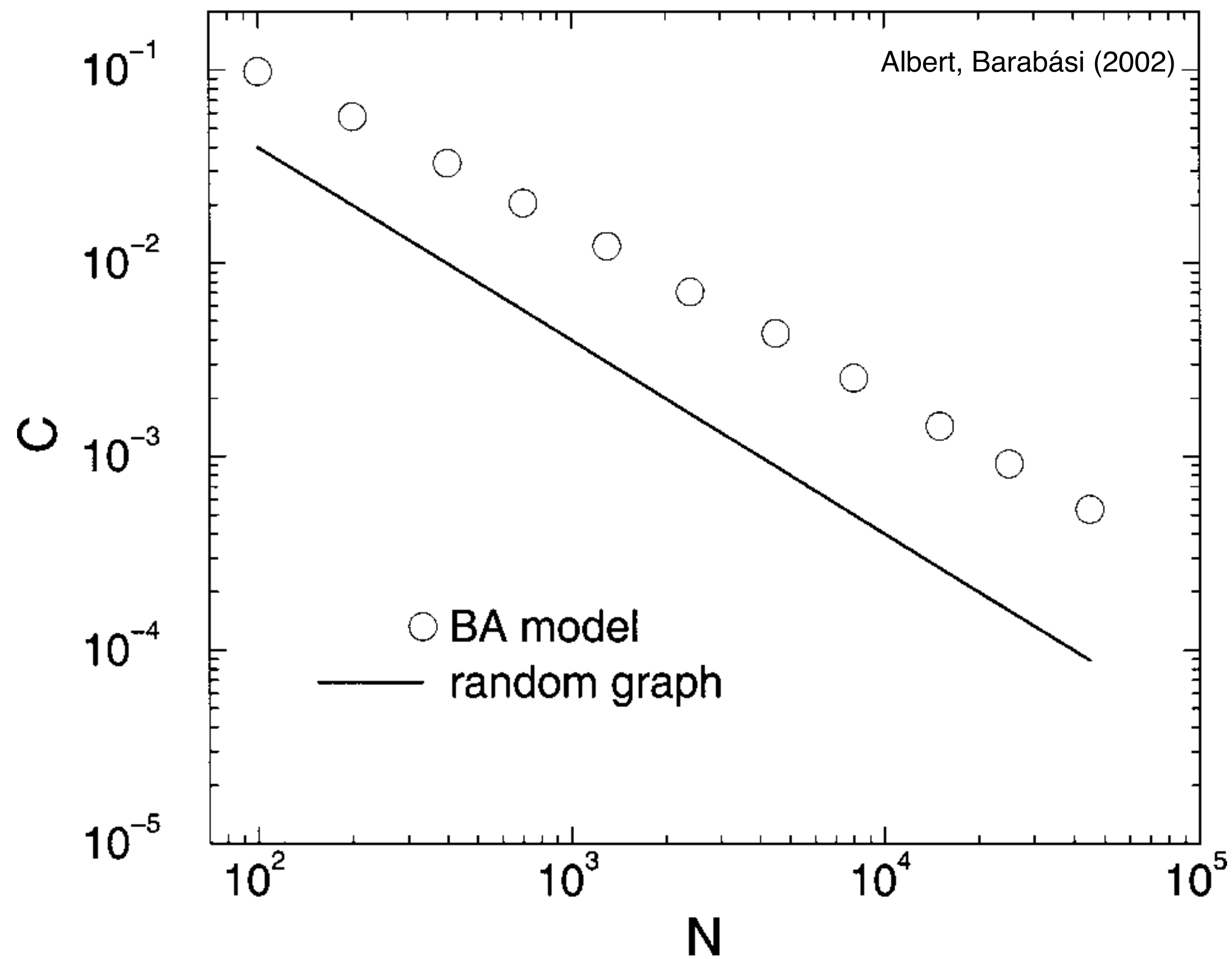
Albert, Barabási (2002)

The BA model - clustering coefficient

- The clustering coefficient decreases with the system size as

$$C = \frac{m}{4}\frac{(\ln N)^2}{N}$$

- It is 5 times more than for random graphs



Degree correlations:

- The BA model is inducing non-trivial degree correlations due to its definition

$$n_{kl} \simeq k^{-2} l^{-2}$$

ER Random Network - catch up

Network	Degree distribution	Path length	Clustering coefficient
Real world networks	broad	short	large
Regular lattices	constant	long	large
ER random networks	Poissonian	short	small
WS small-world networks	exponential	short	large
BA scale-free networks	power-law	short	Rather small

(some)

Other random models

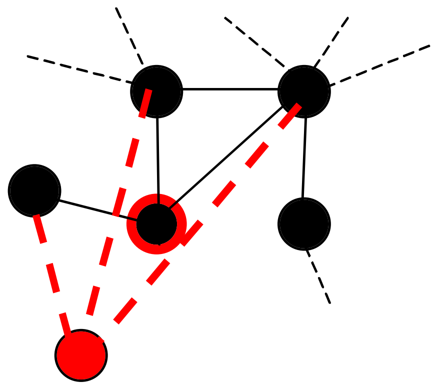
Other scale-free models

The vertex-copying model

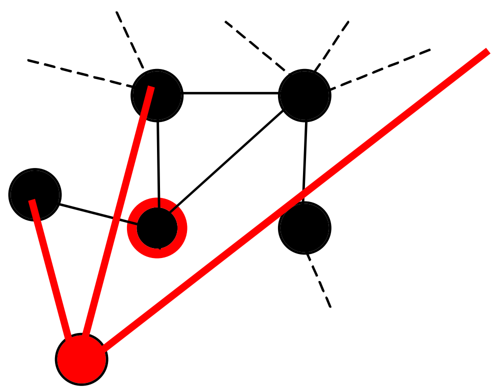
- Motivation:
  - Citations network or WWW where links are often copied
  - Local explanation to preferential attachement

1. Take a small seed network
2. Pick a random vertex
3. Make a copy of it
4. With probability p, move each edge of the copy to point to a random vertex
5. Repeat 2.-4. until the network has grown to desired size of N vertices

1. copy a vertex



2. rewire edges with *p*



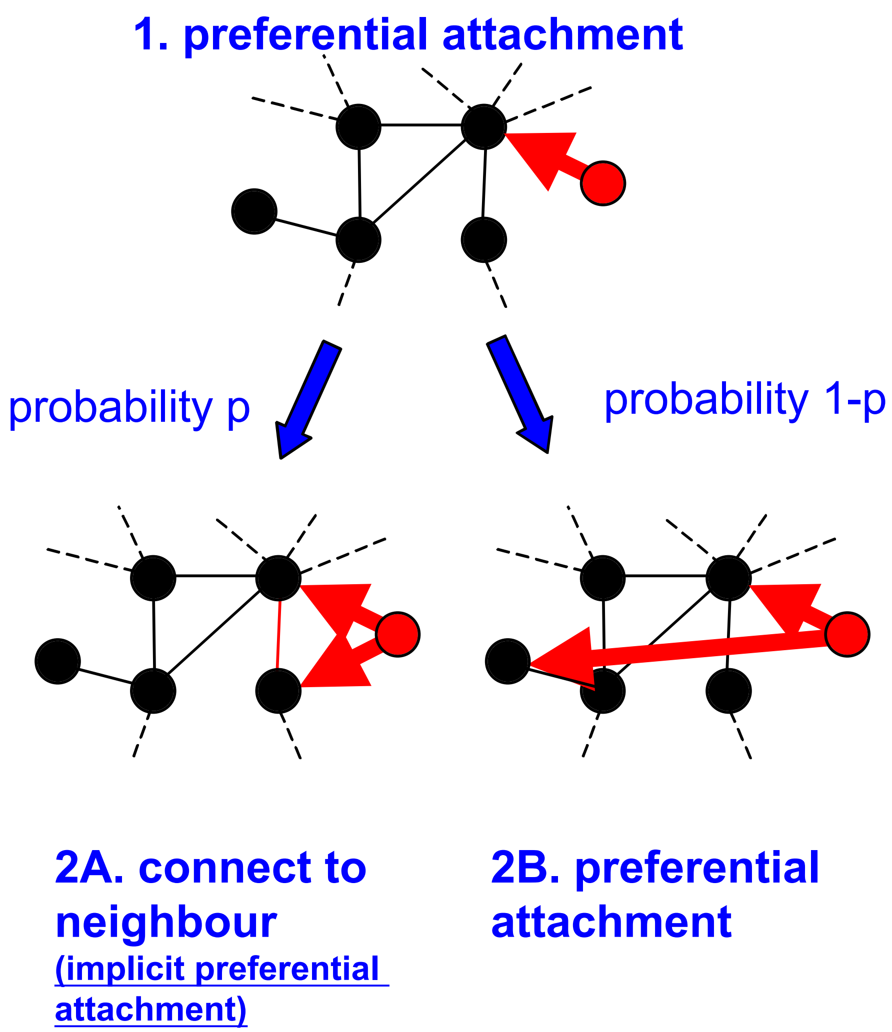
- Asymptotically scale-free with exponent $\gamma \geq 3$

Other scale-free models

The Holme-Kim model

- Motivation: more realistic clustering coefficient

1. Take a small seed network
2. Create a new vertex with m edges
3. Connect the first of the m edges to existing vertices with a probability proportional to their degree k (just like BA)
4. With probability p, connect the next edge to a random neighbour of the vertex of step 3., otherwise do 3. again
5. Repeat 2.-4. until the network has grown to desired size of N vertices



$$C\left(k\right) \propto \frac{1}{k}$$

for large N, ie clustering more realistic! This type of clustering is found in many real-world networks.

Other scale-free models

The forest fire model

- Models generating power laws are usually based on the idea of a growing network.
- It has been shown that real growing networks have two properties: growing average degree, shrinking average shortest path

[Graphs over Time: Densification Laws, Shrinking Diameters and Possible Explanations, leskovec et al. 2012]

Other scale-free models

The forest fire model

Consider a node v joining the network at time $t > 1$, and let G_t be the graph constructed thus far. (G_1 will consist of just a single node.) Node v forms out-links to nodes in G_t according to the following process.

(i) v first chooses an *ambassador node* w uniformly at random, and forms a link to w.

(ii) We generate a random number x that is binomially distributed with mean $(1-p)^{-1}$. Node v selects x links incident to w, choosing from among both out-links and in-links, but selecting in-links with probability r times less than out-links. Let $w_1, w_2, \ldots, w_x$ denote the other ends of these selected links.

(iii) v forms out-links to $w_1, w_2, \ldots, w_x$, and then applies step (ii) recursively to each of $w_1, w_2, \ldots, w_x$. As the process continues, nodes cannot be visited a second time, preventing the construction from cycling.

[Graphs over Time: Densification Laws, Shrinking Diameters and Possible Explanations, leskovec et al. 2012]

Other scale-free models

The forest fire model

• *Heavy-tailed in-degrees.* Our model has a "rich get richer" flavor: highly linked nodes can easily be reached by a newcomer, no matter which ambassador it starts from.

• *Communities.* The model also has a "copying" flavor: a newcomer copies several of the neighbors of his/her ambassador (and then continues this recursively).

• *Heavy-tailed out-degrees.* The recursive nature of link formation provides a reasonable chance for a new node to burn many edges, and thus produce a large out-degree.

• *Densification Power Law.* A newcomer will have a lot of links near the community of his/her ambassador; a few links beyond this, and significantly fewer farther away. Intuitively, this is analogous to the Community Guided Attachment, although without an explicit set of communities.

• *Shrinking diameter.* It is not a priori clear why the Forest Fire Model should exhibit a shrinking diameter as it grows. Graph densification is helpful in reducing the diameter, but it is important to note that densification is certainly not enough on its own to imply shrinking diameter. For example, the Community Guided Attachment model obeys the Densification Power Law, but it can be shown to have a diameter that slowly increases.

Rigorous analysis of the Forest Fire Model appears to be quite difficult.

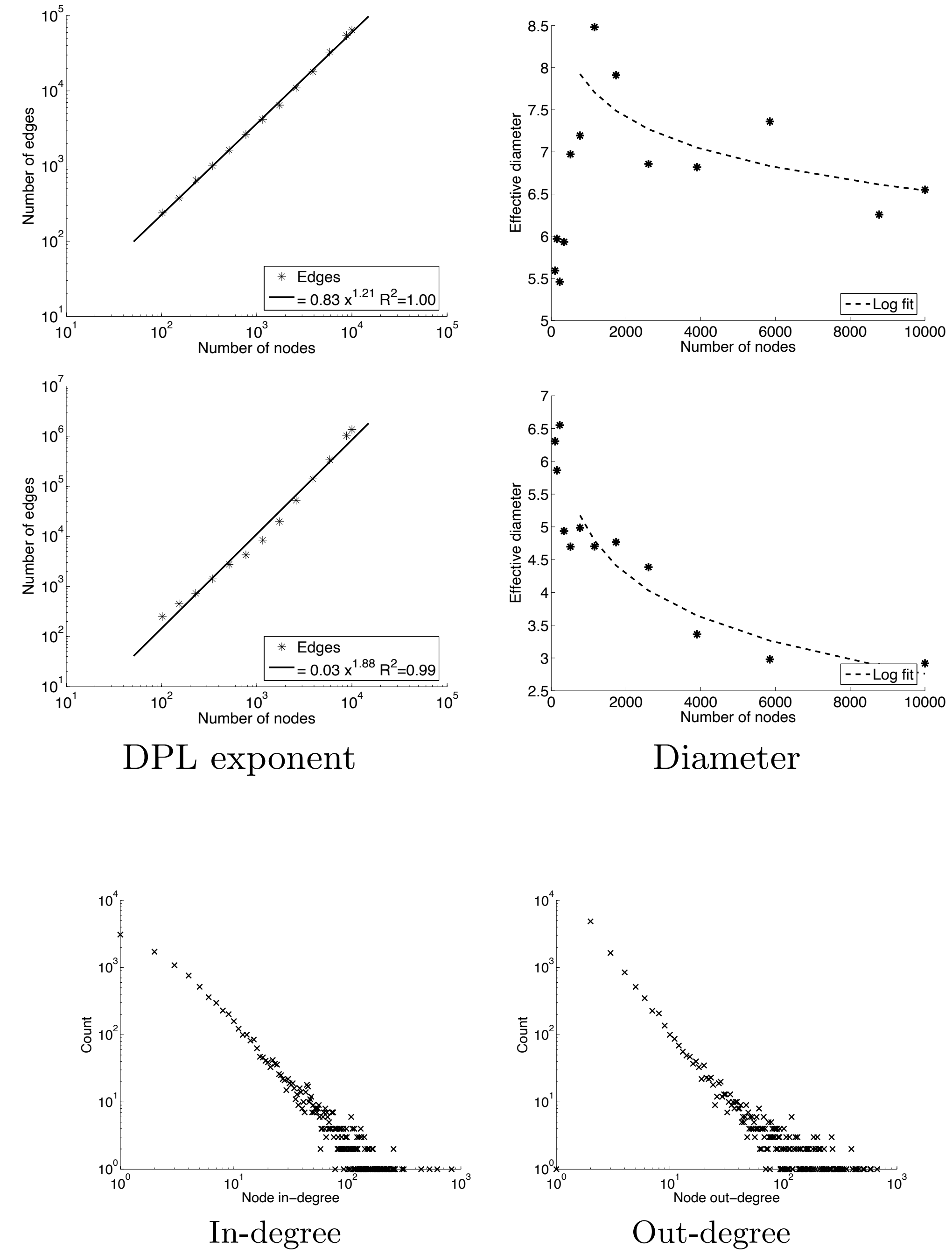


Figure 6: Degree distribution of a sparse graph with decreasing diameter (forward burning probability: 0.37, backward probability: 0.32).

[Graphs over Time: Densification Laws, Shrinking Diameters and Possible Explanations, leskovec et al. 2012]

Other scale-free models

Generalized random models

- Exponential Random Graphs
- Random graphs with latent variables
- Graphons
- …

- Nodes have some **known** properties θ (degrees, attributes…).
- We would like to study **another** property *x*
- We can compare the observed *x* to the distribution of *x* among graphs in the space of all possible graphs respecting θ.
- How to efficiently analyse this graph space ?

ER Random Network - catch up

Network	Degree distribution	Path length	Clustering coefficient
Real world networks	broad	short	large
Regular lattices	constant	long	large
ER random networks	Poissonian	short	small
WS small-world networks	exponential	short	large
BA scale-free networks	power-law	short	Rather small
Other models	power-law	short	Large

Complex models can have all three properties,
But what is the point if they are themselves quite complex?

End notes

- “All models are wrong, but some are useful”
- ER models and Configuration models are used as reference models in a very large number of applications
- WS, BA models are more “making a point” type models: simple processes can explain some non-trivial properties of networks, unfound in random networks.
- Correlation is not causation. Are these simple processes the “cause” ? Maybe, maybe not, sometimes…