Network Science Cheatsheet

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Network Science - Introduction

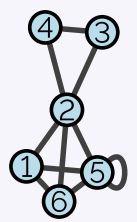
Networks: Graph notation

Graph notation : $G = (V, E)$

V	set of vertices/nodes.
E	set of edges/links.
$u \in V$	a node.
$(u, v) \in E$	an edge.

Network - Graph notation

Graph



Graph notation

$$G = (V, E)$$
$$V = \{1, 2, 3, 4, 5, 6\}$$
$$E = \{(1, 2), (1, 6), (1, 5), (2, 4), (2, 3), (2, 5), (2, 6), (6, 5), (5, 5), (4, 3)\}$$

Types of networks

Simple graph: Edges can only exist or not exist between each pair of node, and there are no self-loops, i.e., an edge connecting a node to itself.
Directed graph: Edges have a direction: $(u, v) \in V$ does not imply $(v, u) \in V$
Weighted graph: A weight is associated to every edge.

Other types of graphs (multigraphs, multipartite, hypergraphs, etc.) are introduced later

Counting nodes and edges

N, n	**size**: number of nodes $\|V\|$.
L, m	number of edges $\|E\|$
L_{max}	Maximum number of links

Undirected network: $\binom{N}{2} = N(N-1)/2$

Directed network: $\binom{N}{2} = N(N-1)$

Node-Edge description

N_u	**Neighbourhood** of u, nodes sharing a link with u.
k_u	**Degree** of u, number of neighbors $\|N_u\|$.
N_u^{out}	**Successors** of u, nodes such as $(u, v) \in E$ in a directed graph
N_u^{in}	**Predecessors** of u, nodes such as $(v, u) \in E$ in a directed graph
k_u^{out}	**Out-degree** of u, number of outgoing edges $\|N_u^{out}\|$.
k_u^{in}	**In-degree** of u, number of incoming edges $\|N_u^{in}\|$
$w_{u,v}$	**Weight** of edge (u, v).
s_u	**Strength** of u, sum of weights of adjacent edges, $s_u = \sum_v w_{uv}$.

Network descriptors - Nodes/Edges

$\langle k \rangle$ — **Average degree**: Real networks are sparse, i.e., typically $\langle k \rangle \ll n$. Increases slowly with network size, e.g., $\langle k \rangle \sim \log(n)^a$

$$\langle k \rangle = \frac{2m}{n}$$

$d, d(G)$ — **Density**: Fraction of pairs of nodes connected by an edge in G.

$$d = L/L_{\max}$$

[a] Leskovec, Kleinberg, and Faloutsos 2005.

Paths - Walks - Distance

Walk: Sequences of adjacent edges or nodes (e.g., **1,2,1,6,5** is a valid walk)
Path: a walk in which each node is distinct.
Path length: number of **edges** encountered in a path
Weighted Path length: Sum of the weights of edges on a path
Shortest path: The shortest path between nodes u, v is a path of minimal *path length*. Not necessarily unique.
Weighted Shortest path: path of minimal *weighted path length*.
$\ell_{u,v}$: **Distance**: The distance between nodes u, v is the length of the shortest path

Network descriptors - Paths

$\ell_{\max}$ — **Diameter**: maximum *distance* between any pair of nodes.

$\langle \ell \rangle$ — **Average distance**:

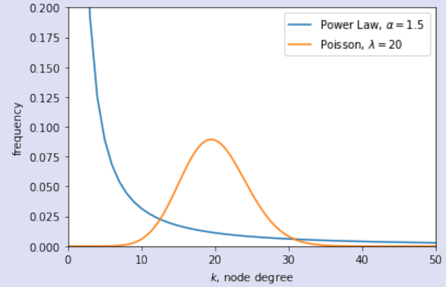
$$\langle \ell \rangle = \frac{1}{n(n-1)} \sum_{i \neq j} \ell_{ij}$$

Degree distribution

The degree distribution is considered an important network property. They can follow two typical distributions:

- **Bell-curved** shaped (Normal/Poisson/Binomial)
- **Scale-free**, also called *Power-law*

A Bell-curved distribution has a *typical scale*: as human height, it is centered on an average value. A Scale-free distribution has no typical scale: as human wealth, its average value is not representative, low values (degrees) are the most frequent, while a few very large values can be found (hubs, large degree nodes). It has a *long tail*, meaning that rare (large) values are not as rare as in a bell-curved distribution.



More details later.

Subgraphs

Subgraph $H(W)$ (induced subgraph): subset of nodes W of a graph $G = (V, E)$ and edges connecting them in G, i.e., subgraph $H(W) = (W, E'), W \subset V, (u, v) \in E' \iff u, v \in W \wedge (u, v) \in E$
Clique: subgraph with $d = 1$
Triangle: clique of size 3
Connected component: a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertex in the supergraph
Strongly Connected component: In directed networks, a subgraph in which any two vertices are connected to each other by paths
Weakly Connected component: In directed networks, a subgraph in which any two vertices are connected to each other by paths if we disregard directions

Triangles counting

δ_u - **Triads of** u: number of triangles containing node u
Δ - **Number of triangles in the graph** total number of triangles in the graph, $\Delta = \frac{1}{3}\sum_{u \in V} \delta_u$.

> Each **triangle** in the graph is counted as a **triad** once by each of its nodes.

$\delta_u^{\max}$ - **Triad potential of** u: maximum number of triangles that could exist around node u, given its degree: $\delta_u^{\max} = \binom{k_u}{2}$
$\Delta^{\max}$ - **Triangle potential of G:** maximum number of triangles that could exist in the graph, given its degree distribution: $\Delta^{\max} = \frac{1}{3}\sum_{u \in V} \delta^{\max}(u)$

Clustering Coefficents - Triadic closure

The clustering coefficient is a measure of the triadic closure of a network or of a node neighborhood. The triadic closure is a notion coming from social network analysis, often summarized by the aphorism *The friends of my friends are my friends*.

C_u - **Node clustering coefficient:** density of the subgraph induced by the neighborhood of u, $C_u = d(H(N_u))$. Also interpreted as the fraction of all possible triangles in N_u that exist, $\frac{\delta_u}{\delta_u^{\max}}$
$\langle C \rangle$ - **Average clustering coefficient:** Average clustering coefficient of all nodes in the graph, $\bar{C} = \frac{1}{N}\sum_{u \in V} C_u$.

> Be careful when interpreting this value, since all nodes contributes equally, irrespectively of their degree, and that low degree nodes tend to be much more frequent than hubs, and their C value is very sensitive, i.e., for a node u of degree 2, $C_u \in \{0, 1\}$, while nodes of higher degrees tend to have more contrasted scores.

C^g - **Global clustering coefficient:** Fraction of all possible triangles in the graph that do exist, $C^g = \frac{\Delta}{\Delta^{\max}}$

Small World Network

A network is said to have the **small world** property when it has some structural properties[a]. The notion is usually not quantitatively defined, but two properties are required:

- Average distance must be short, i.e., $\langle \ell \rangle \approx \log(N)$
- Clustering coefficient must be high, i.e., much larger than in a random network , e.g., $C^g \gg d$, with d the network density

This property is considered characteristic of *real* networks, as opposed to random networks. It is believed to be associated to particular properties (robustness to failures, efficient information flow, etc.), and to be the consequence of emergent mechanisms typical of *complex systems*.

Be careful: in some contexts, *small world network* can be used for a network that has a small Average distance, without considering its Clustering Coefficient.

[a] Watts and Strogatz 1998.

Cores and Shells

Many real networks are known to have a **core-periphery** structure, i.e., there is a densely connected core at its center and a more peripheral zone in which nodes are loosely connected between them and to the core.

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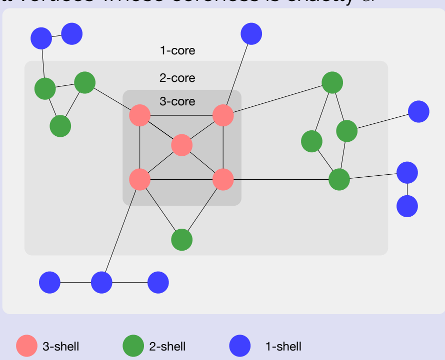
k-core: The k-core (core of order k) of $G(V, E)$ is the largest subgraph $H(C)$ such as all nodes have at least a degree k, i.e., $\forall u \in C, k_u^H \geq k$, with k_u^H the degree of node u in subgraph H.
coreness: A vertex u has coreness k if it belongs to the k-core but not to the $k+1$-core.
c-shell: all vertices whose coreness is exactly c.



Vocabulary

Singleton: node with a degree $k = 0$
Hub: node u with $k_u \gg \langle k \rangle$

Bridge: Edge which, when removed, split a connected component in two.
Self-loop: Edge which connects a node to itself.
Stub: A stub is an half edge, i.e., edge (u, v) has a stub connected to u and another connected to v.

Complete network: $L = L_{max}$
Sparse network: $d \ll 1, L \ll L_{max}$
Connected Graph: Graph composed of a single connected component

Going Further

Books about network science as a whole:

- Barabási et al. 2016 (free)
- Coscia 2021 (free)
- Zinoviev 2018
- Menczer, Fortunato, and Davis 2020

References

[1] Albert-László Barabási et al. *Network science*. Cambridge university press, 2016.

[2] Michele Coscia. *The Atlas for the Aspiring Network Scientist*. 2021. arXiv: `2101.00863 [cs.CY]`.

[3] Jure Leskovec, Jon Kleinberg, and Christos Faloutsos. "Graphs over time: densification laws, shrinking diameters and possible explanations". In: *Proceedings of the eleventh ACM SIGKDD international conference on Knowledge discovery in data mining*. 2005, pp. 177–187.

[4] Filippo Menczer, Santo Fortunato, and Clayton A Davis. *A First Course in Network Science*. Cambridge University Press, 2020.

[5] Duncan J Watts and Steven H Strogatz. "Collective dynamics of 'small-world'networks". In: *nature* 393.6684 (1998), pp. 440–442.

[6] Dmitry Zinoviev. *Complex network analysis in Python: Recognize-construct-visualize-analyze-interpret*. Pragmatic Bookshelf, 2018.