COMPLEX NETWORKS

WHO AM I

- Rémy Cazabet
- Associate Professor (Maître de conférences)
  - Université Lyon 1
  - LIRIS, DM2L Team (Data Mining & Machine Learning)
- Computer Scientist => Network Scientist
- Member of IXXI

RESOURCES

- Website of the course:
  - http://cazabetremy.fr/Teaching/ComplexNetworks.html
  - Slides, Cheat sheets, notebooks, etc.

- Contact me: remy.cazabet@univ-lyon1.fr

- I don't have a way to contact you:
  - Please send an email to the address above with: 1)your name, 2)the master you are in (Physics, Computer science, Cognitive science, etc.)

E-LEARNING

- All classes can be followed in streaming at:
  - https://univ-lyon1.webex.com/meet/remy.cazabet
  - (Avoid using Wifi in the room during the class…)
- Recording of classes will be available
- Discord channel, join with: https://discord.gg/VpmZre2
  - Ask questions that can be helpful to others, about exams, difficult points, etc.

CLASS OVERVIEW

- Lectures: 24h
- Tutorials (TD)
  - 3x2h
  - Lorenza Pacini
- Evaluation:
  - Lectures: Writing exam
  - Tutorials: projects during semester

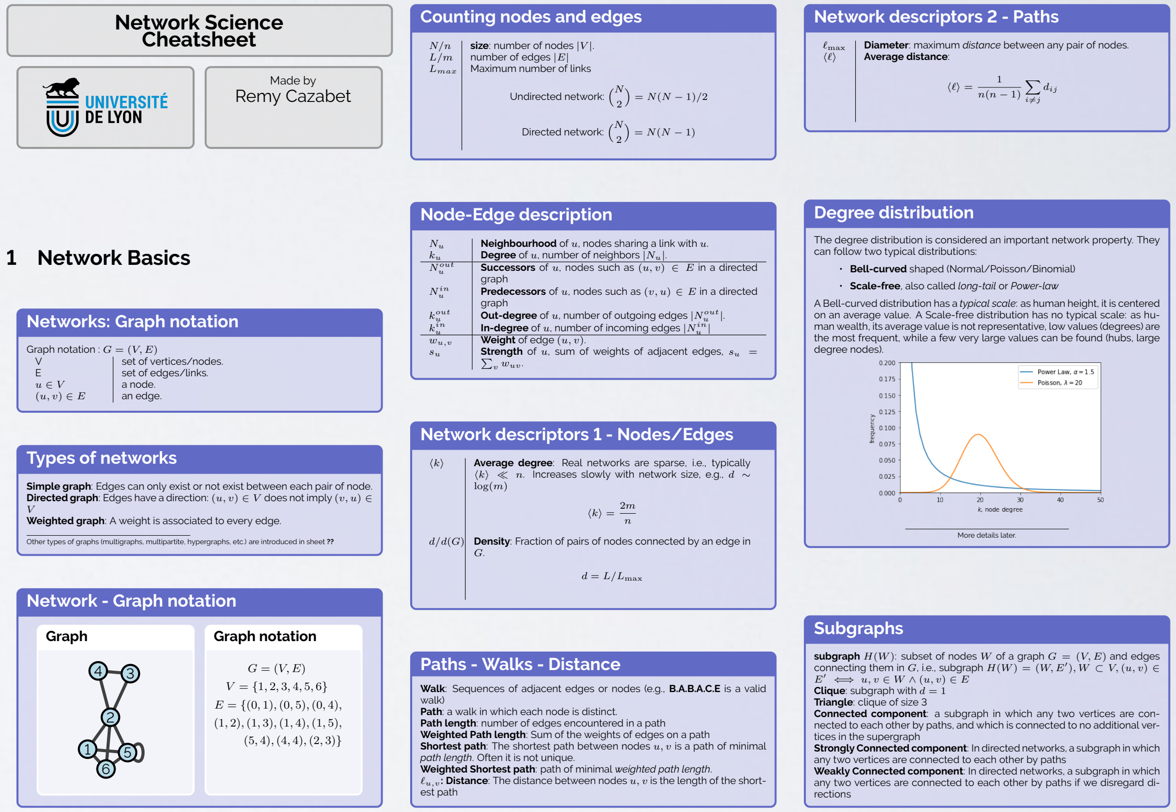
LECTURES

- From next session:
  - 1st half: Theory, me talking on slides
  - 2nd half: You experimenting on computers
    - Please try to bring a computer with battery, barely any power in the room…
    - I'll try to bring a multi-socket adaptor, but only for 4 or 5 computers…

- Please install on your computer:
    - Gephi: https://gephi.org software to manipulate and visualize networks
    - Python, and some libraries: networkx, sklearn, seaborn (for now) cdlib, tnetwork (for later)
    - Also for python: Jupyter notebook.
    - In case of problems with your computer, all the python work can also be done using google colab (https://colab.research.google.com) an online python notebook.

LECTURES

- Don't take definitions, etc.
  - Slides, Cheatsheet, etc.



Network Science Cheatsheet

UNIVERSITÉ DE LYON

Made by
Remy Cazabet

1 Network Basics

Networks: Graph notation

Graph notation : $G = (V, E)$

V	set of vertices/nodes.
E	set of edges/links.
$u \in V$	a node.
$(u, v) \in E$	an edge.

Types of networks

Simple graph: Edges can only exist or not exist between each pair of node.
Directed graph: Edges have a direction: $(u, v) \in V$ does not imply $(v, u) \in V$
Weighted graph: A weight is associated to every edge.

Other types of graphs (multigraphs, multipartite, hypergraphs, etc.) are introduced in sheet ??

Network - Graph notation

Graph

Graph notation

$$G = (V, E)$$
$$V = \{1, 2, 3, 4, 5, 6\}$$
$$E = \{(0,1), (0,5), (0,4), (1,2), (1,3), (1,4), (1,5), (5,4), (4,4), (2,3)\}$$

Counting nodes and edges

N/n	**size**: number of nodes $\lvert V \rvert$.
L/m	number of edges $\lvert E \rvert$
L_{max}	Maximum number of links

Undirected network: $\binom{N}{2} = N(N-1)/2$

Directed network: $\binom{N}{2} = N(N-1)$

Node-Edge description

N_u	**Neighbourhood** of u, nodes sharing a link with u.
k_u	**Degree** of u, number of neighbors $\lvert N_u \rvert$.
N_u^{out}	**Successors** of u, nodes such as $(u, v) \in E$ in a directed graph
N_u^{in}	**Predecessors** of u, nodes such as $(v, u) \in E$ in a directed graph
k_u^{out}	**Out-degree** of u, number of outgoing edges $\lvert N_u^{out} \rvert$.
k_u^{in}	**In-degree** of u, number of incoming edges $\lvert N_u^{in} \rvert$
$w_{u,v}$	**Weight** of edge (u, v).
s_u	**Strength** of u, sum of weights of adjacent edges, $s_u = \sum_v w_{uv}$.

Network descriptors 1 - Nodes/Edges

$\langle k \rangle$ — **Average degree**: Real networks are sparse, i.e., typically $\langle k \rangle \ll n$. Increases slowly with network size, e.g., $d \sim \log(m)$

$$\langle k \rangle = \frac{2m}{n}$$

$d/d(G)$ — **Density**: Fraction of pairs of nodes connected by an edge in G.

$$d = L/L_{\max}$$

Paths - Walks - Distance

Walk: Sequences of adjacent edges or nodes (e.g., **B.A.B.A.C.E** is a valid walk)
Path: a walk in which each node is distinct.
Path length: number of edges encountered in a path
Weighted Path length: Sum of the weights of edges on a path
Shortest path: The shortest path between nodes u, v is a path of minimal *path length*. Often it is not unique.
Weighted Shortest path: path of minimal *weighted path length*.
$\ell_{u,v}$: **Distance**: The distance between nodes u, v is the length of the shortest path

Network descriptors 2 - Paths

$\ell_{\max}$ — **Diameter**: maximum *distance* between any pair of nodes.
$\langle \ell \rangle$ — **Average distance**:

$$\langle \ell \rangle = \frac{1}{n(n-1)} \sum_{i \neq j} d_{ij}$$

Degree distribution

The degree distribution is considered an important network property. They can follow two typical distributions:

- **Bell-curved** shaped (Normal/Poisson/Binomial)
- **Scale-free**, also called *long-tail* or *Power-law*

A Bell-curved distribution has a *typical scale*: as human height, it is centered on an average value. A Scale-free distribution has no typical scale: as human wealth, its average value is not representative, low values (degrees) are the most frequent, while a few very large values can be found (hubs, large degree nodes).

More details later.

Subgraphs

subgraph $H(W)$: subset of nodes W of a graph $G = (V, E)$ and edges connecting them in G, i.e., subgraph $H(W) = (W, E'), W \subset V, (u, v) \in E' \iff u, v \in W \wedge (u, v) \in E$
Clique: subgraph with $d = 1$
Triangle: clique of size 3
Connected component: a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the supergraph
Strongly Connected component: In directed networks, a subgraph in which any two vertices are connected to each other by paths
Weakly Connected component: In directed networks, a subgraph in which any two vertices are connected to each other by paths if we disregard directions

COMPLEX NETWORKS

WHAT?
WHY?
WHY NOW?
WHAT FOR?

SCIENCE

- Science: understanding how things work
  - The human body, the motion/characteristics of objects, societies, etc.
- Step 1 of science (historically): Experiment with the object (macro-level)
  - What if I throw a ball from that height ? From a moving platform ? If it’s a dice ? In wood or in glass?
  - What if I give this substance to eat/drink? Is sickness related to cold? Humidity? etc.

SCIENCE

- 2)Great success of the 19/20 centuries: **Reductionism**
- To understand things, I need to understand what they are made of:
  - A human body: organs, vessels => cells => DNA, proteins & stuff => Nucleotides ….
  - Objects: Organic compounds => atoms => protons/electrons/neutrons => stuff
- => Now we know. And then what ?

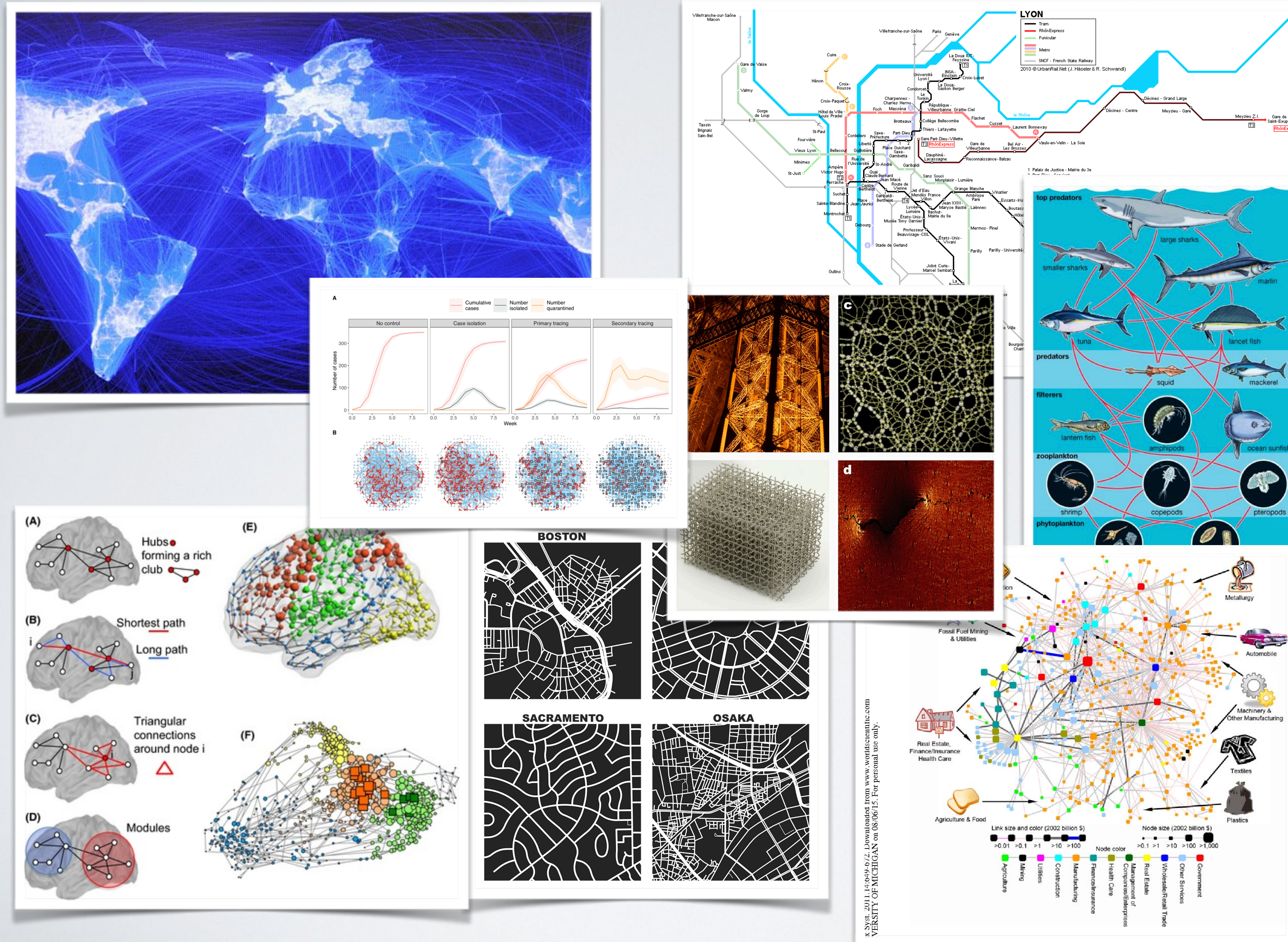
SCIENCE

- 3) Two situations:
  - The system is **homogeneous** and/or has a **regular** structure
    - => You can explain it with a bunch of equations
  - The system is **heterogeneous** and/or **has a complex structure**
    - => Understanding each component is not enough to understand the system
    - Understanding each cell tells you little about how the brain works.
    - Understanding how each individual works/behave tells you little about societies
    - etc.

- => The structure/relations/interactions matters.
  - Networks represent structures

COMPLEX SYSTEMS

- **Complex systems**: Systems composed of multiple **parts** in **interactions**
- Complex networks model the interactions between the parts
  - A common framework applicable to many systems
  - =>Many networks share similar characteristics
  - =>Similar processes shape the networks



LYON
Tram
RhônExpress
Funicular
Metro
SNCF - French State Railway
2010 © UrbanRail.Net (J. Häseler & R. Schwandl)
A
Cumulative cases
Number isolated
Number quarantined
No control
Case isolation
Primary tracing
Secondary tracing
Number of cases
Week
B
c
d
top predators
large sharks
smaller sharks
marlin
tuna
lancet fish
predators
squid
mackerel
filterers
lantern fish
amphipods
ocean sunfish
zooplankton
shrimp
copepods
pteropods
phytoplankton
(A)
Hubs forming a rich club
(B)
Shortest path
Long path
(C)
Triangular connections around node i
(D)
Modules
(E)
(F)
BOSTON
SACRAMENTO
OSAKA
Metallurgy
Fossil Fuel Mining & Utilities
Automobile
Machinery & Other Manufacturing
Real Estate, Finance/Insurance Health Care
Textiles
Plastics
Agriculture & Food
Link size and color (2002 billion $)
Node size (2002 billion $)
Node color
Agriculture
Mining
Utilities
Construction
Manufacturing
Finance/Insurance
Health Care
Management of Companies/Enterprises
Real Estate
Wholesale/Retail Trade
Other Services
Government

WHO ?

- Network scientists:
  - Physicists
  - Computer scientists
  - Mathematicians
  - Sociologists
  - => Work on the same problems, with converging vocabularies and references
- Applied network scientists
  - Geographers, biologists, social scientists, economists, etc.
  - =>Experts of i)their domain, and ii)complex networks analysis

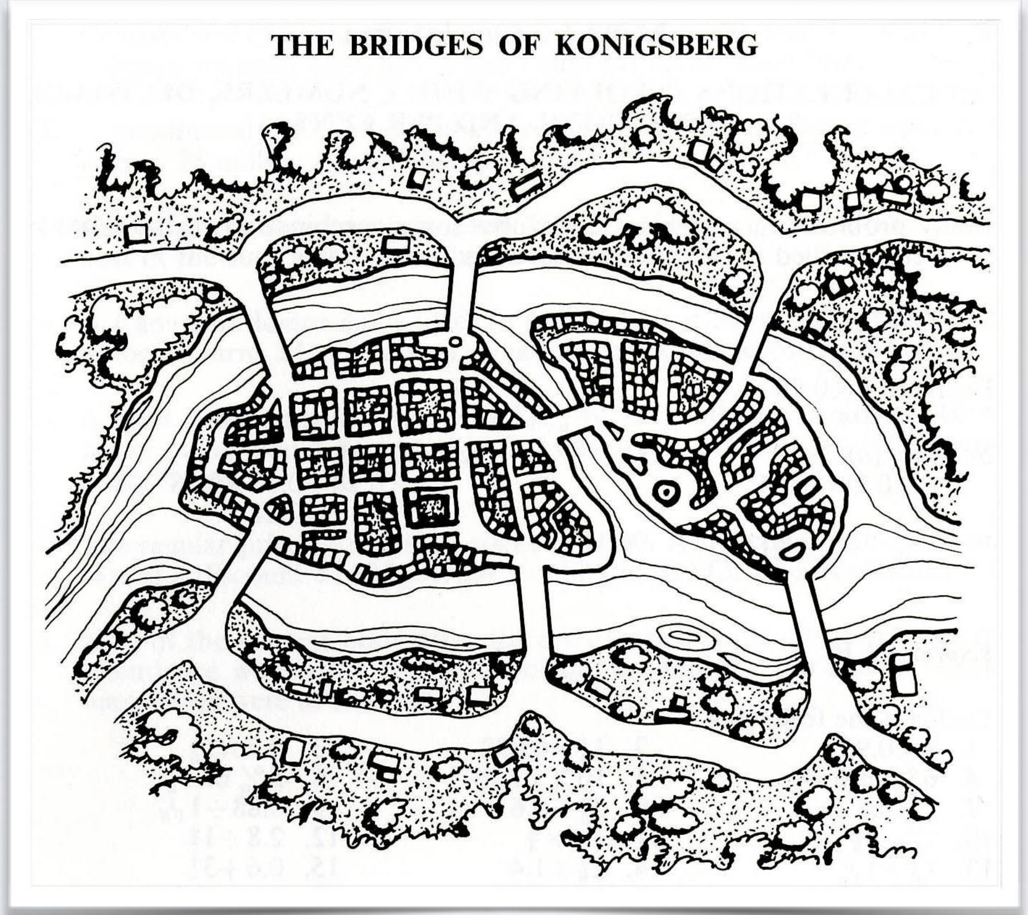
TO CONCLUDE

- Complex Network Analysis *is/should be/will become* (in my opinion) one of the basic tools of the modern scientist (and Data scientist), much as *statistics* or *linear algebra*.

A BRIEF HISTORY

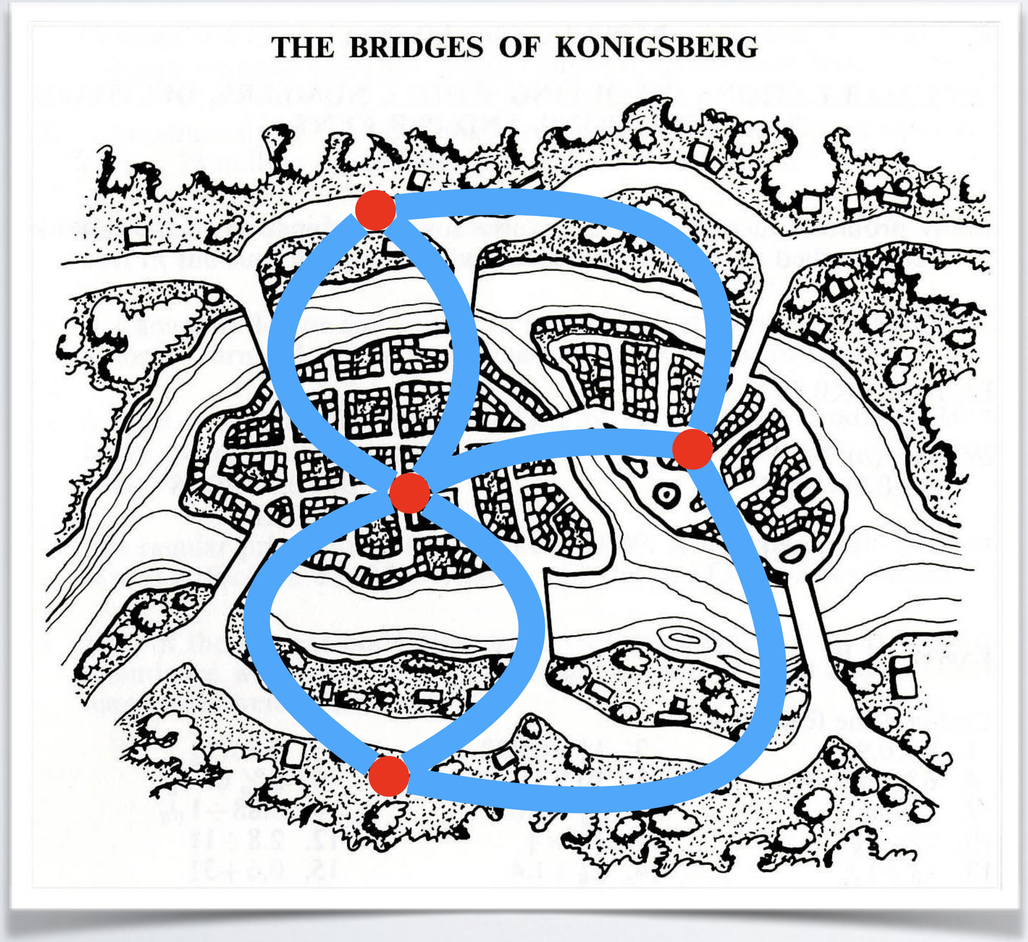
A BRIEF HISTORY

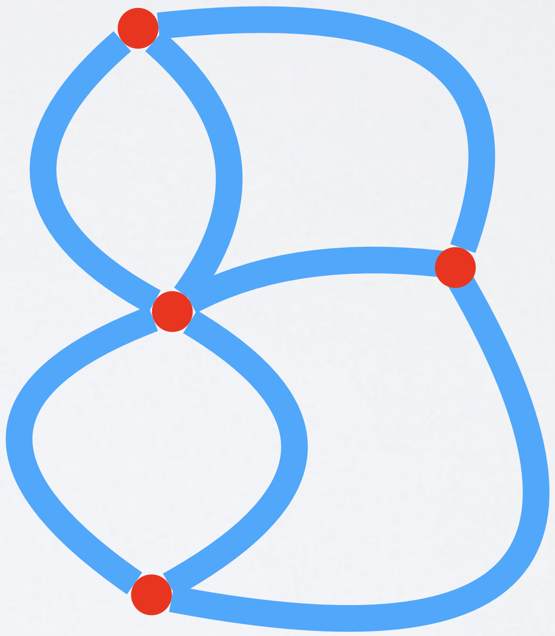
- Graph theory:1736 - Euler and the bridges of konigsberg



Can one walk across the seven bridges and never cross the same bridge twice?

A BRIEF HISTORY





Answer: **No**

A BRIEF HISTORY

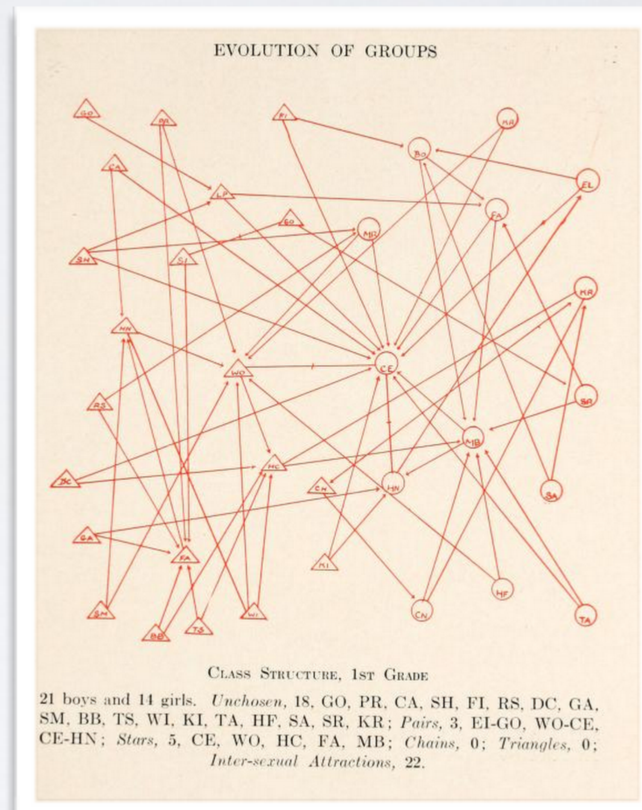
- Social networks: 1934 - Jacob Moreno

CLASS:_____ QUESTION:____________

NOMINEE'S ID NUMBERS

NOMINATOR'S ID NUMBER

SEX	ID #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
F	1	■				+				-	-		+								+	-			
M	2	-	■							+						+				+			-	-	
F	3			■			+	-			-		+				+					-			
F	4				■						-		+	+			+			-	-				
F	5	+				■					-			+	+					-	-				
F	6	-		+			■						+			-	+						-		
M	7		+					■			-	+								-		-		+	
F	8				+		-		■					+				-		+					-
M	9		+					+		■		+		-		-									-
M	10		+					-			■	-				+				+		-			
M	11		+								-	■				+		+		-	-				
F	12	+						-			-		■			-	+				+				
F	13				+								+	■			+			-	-	-			
F	14					+	-	+		-					■						+		-		
M	15							+			-				-	■				+				+	-
F	16				+						-		+				■				+		-	-	
M	17				-							+						■		+	-	-		+	
M	18							-				+							■	-		-	+		+
M	19		-									+	-			+				■	+	-			
F	20						-			-			+		-		+			+	■				
F	21	-	-		+								+							-	+	■			
M	22						-			-		+					-	+					■		+
M	23	-						+				+						-		+		-		■	
M	24											+						+			-	-	+	-	■
TOTAL	+	2	4	1	5	2	1	4	0	1	0	8	8	3	1	4	6	3	0	7	6	0	2	3	2
TOTAL	-	4	2	0	1	0	4	4	0	4	9	1	1	1	2	3	1	2	0	7	6	10	4	3	3



Sociomatrix

Sociograms

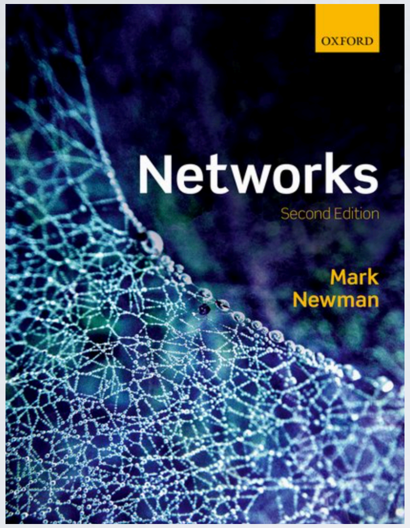
KEY PUBLICATIONS

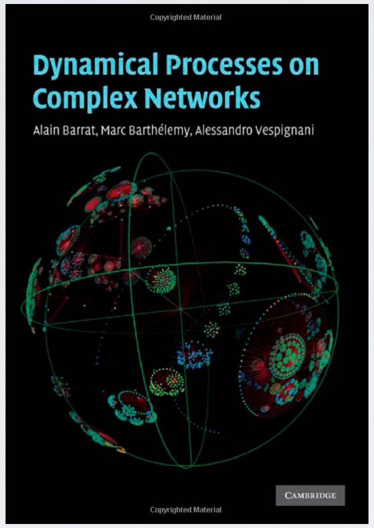
- 1998: Watts & Strogatz - Small-World:
  - 2nd Most cited paper of the year in Nature
- 1999: Barabasi & Albert - scale-free networks:
  - Most cited paper of the year in Science
- 2002: Girvan & Newman - Community detection:
  - Most cited paper of the year in PNAS
- 2004: Barabasi & Oltvai - Network Biology:
  - Most cited paper (ever) in Nature genetics
- 2010: Kwak et al. - What is Twitter, a Social Network or a News Media?
  - Most cited paper (ever) of the WWW conference
- …

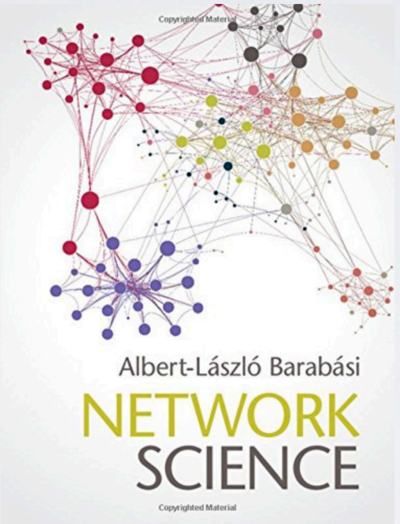
(As of 2020)

Materials

Lecture books







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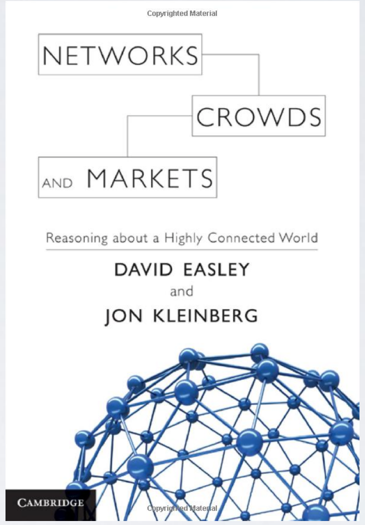

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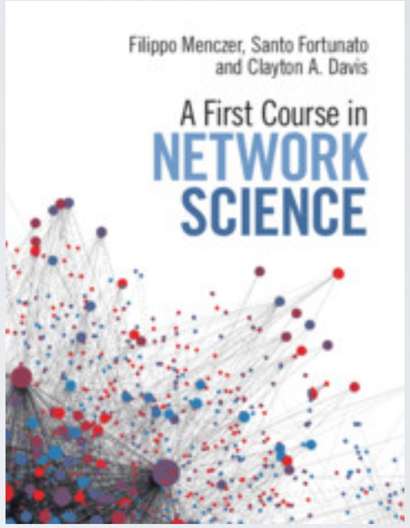

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Reviews

SIAM Review
Vol. 45, No. 2, pp. 167–256

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The Structure and Function of Complex Networks*

M. E. J. Newman†

REVIEWS OF MODERN PHYSICS, VOLUME 74, JANUARY 2002

Statistical mechanics of complex networks

Réka Albert* and Albert-László Barabási

Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556

Characterization and Modeling of weighted networks

Marc Barthélemy[1], Alain Barrat[2], Romualdo Pastor-Satorras[3], and Alessandro Vespignani[2]

Physics Reports 486 (2010) 75–174

Contents lists available at ScienceDirect

Physics Reports

journal homepage: www.elsevier.com/locate/physrep

ELSEVIER

Community detection in graphs

Santo Fortunato*

Complex Networks and Systems Lagrange Laboratory, ISI Foundation, Viale S. Severo 65, 10133, Torino, I, Italy

Physics Reports 519 (2012) 97–125

Contents lists available at SciVerse ScienceDirect

Physics Reports

journal homepage: www.elsevier.com/locate/physrep

ELSEVIER

Temporal networks

Petter Holme[a,b,c,*], Jari Saramäki[d]

[a] IceLab, Department of Physics, Umeå University, 901 87 Umeå, Sweden
[b] Department of Energy Science, Sungkyunkwan University, Suwon 440–746, Republic of Korea
[c] Department of Sociology, Stockholm University, 106 91 Stockholm, Sweden
[d] Department of Biomedical Engineering and Computational Science, School of Science, Aalto University, 00076 Aalto, Espoo, Finland

Contents lists available at ScienceDirect

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journal homepage: www.elsevier.com/locate/physrep

ELSEVIER

Spatial networks

Marc Barthélemy*

Contents lists available at ScienceDirect

Physics Reports

journal homepage: www.elsevier.com/locate/physrep

ELSEVIER

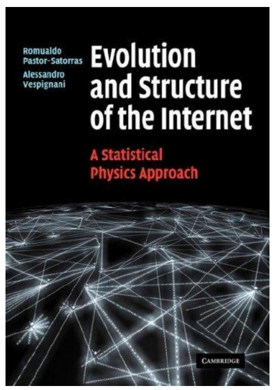
The structure and dynamics of multilayer networks

S. Boccaletti[a,b,*], G. Bianconi[c], R. Criado[d,e], C.I. del Genio[f,g,h], J. Gómez-Gardeñes[i], M. Romance[d,e], I. Sendiña-Nadal[j,e], Z. Wang[k,l], M. Zanin[m,n]

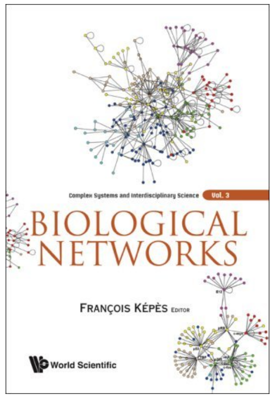
…and many more…all of them on arXiv.org!

Materials

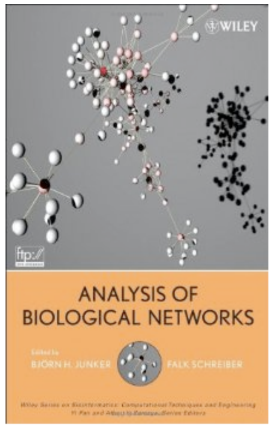
Related books



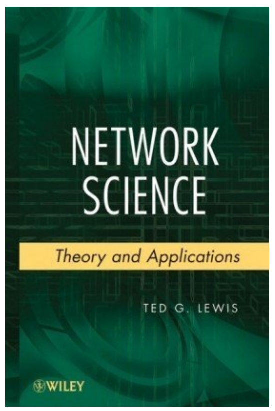
R. Pastor-Satorras, A. Vespignani, Evolution and Structure of the Internet: A Statistical Physics Approach (Cambridge University Press, 2007), rst edn.



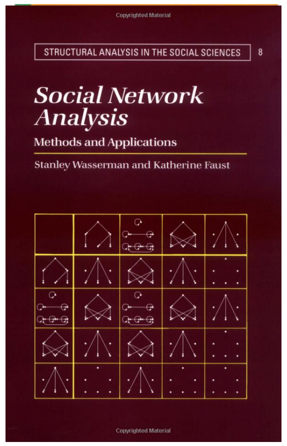
F. Kopos, Biological Networks (Complex Systems and Interdisciplinary Science) (World Scientific Publishing Company, 2007), rst edn.



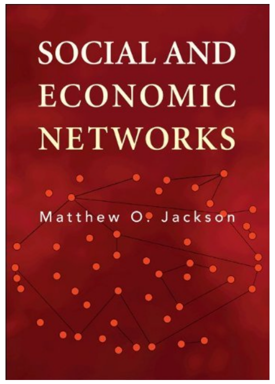
B. H. Junker, F. Schreiber, Analysis of Biological Networks (Wiley Series in Bioinformatics) (Wiley-Interscience, 2008).



T. G. Lewis, Network Science: Theory and Applications (Wiley, 2009).



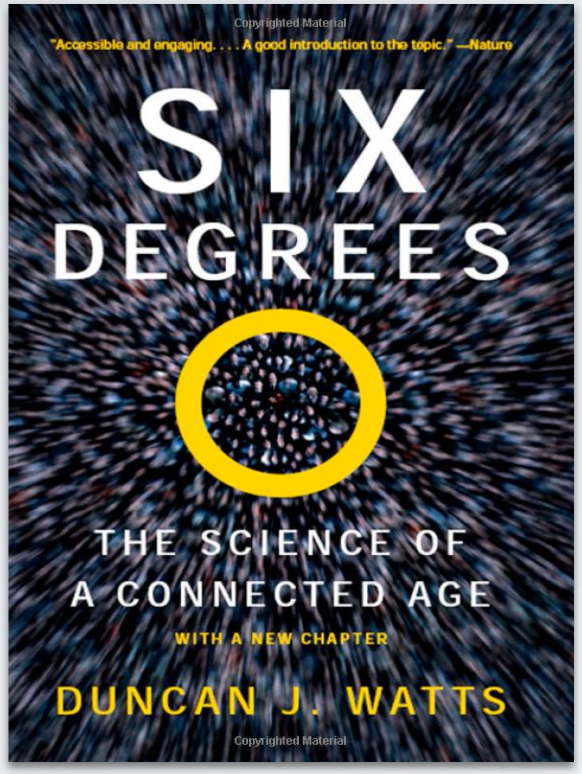
S. Wasserman and K. Faust
Social Network Analysis (Methods and Applications) x
Cambridge University Press (1994)

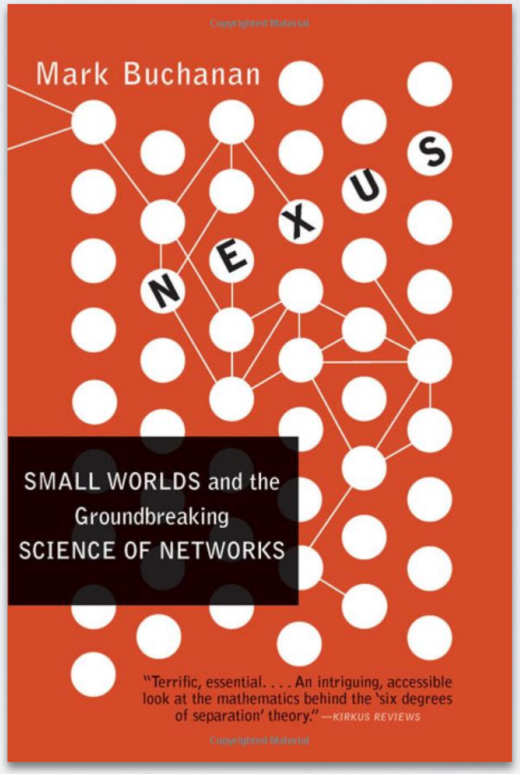


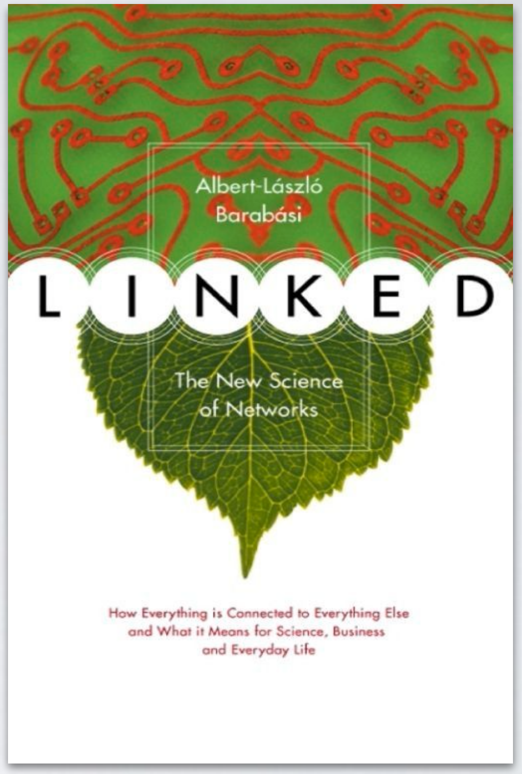
M. O. Jackson, Social and Economic Networks (Princeton University Press, 2010).

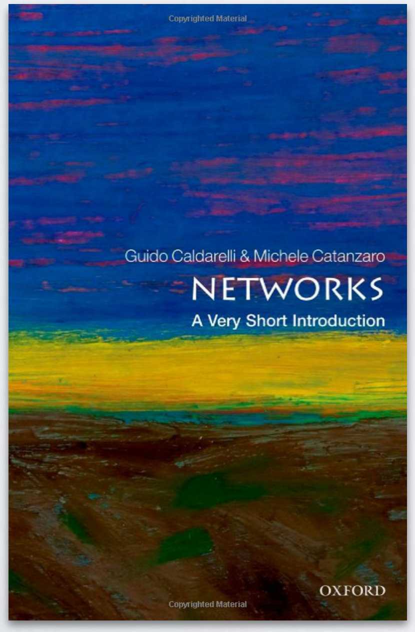
Materials

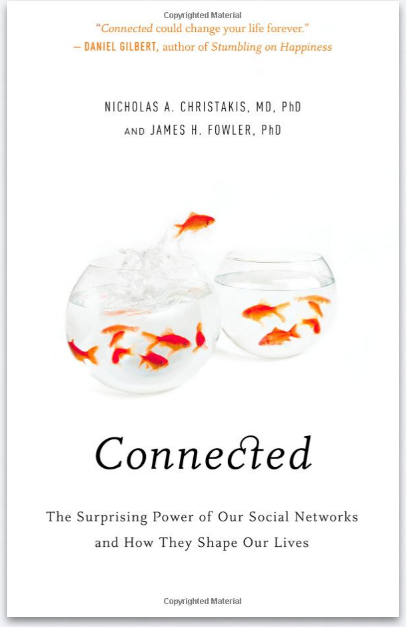
Pop-science books

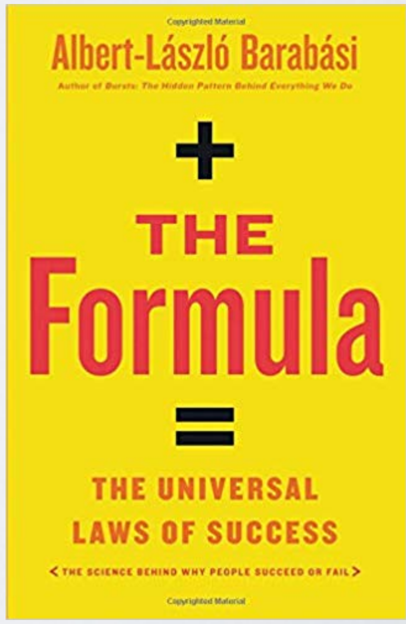












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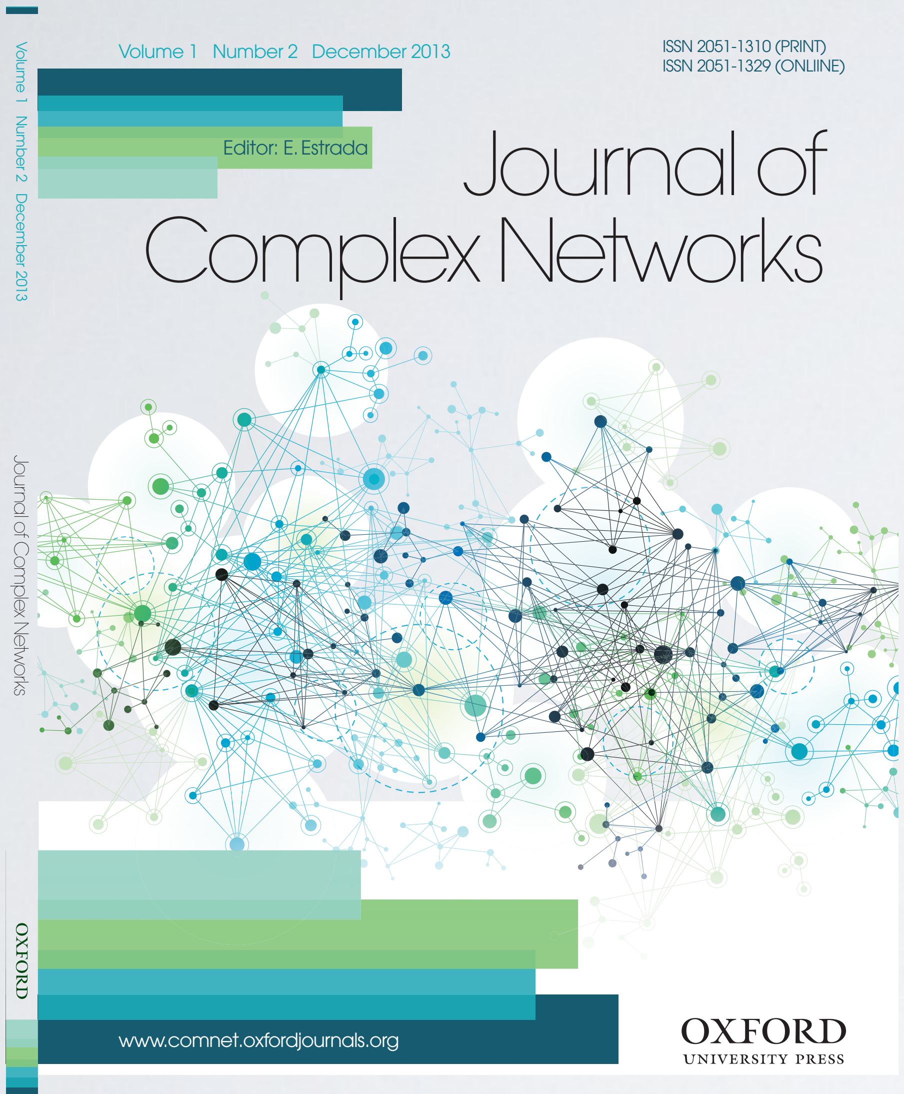
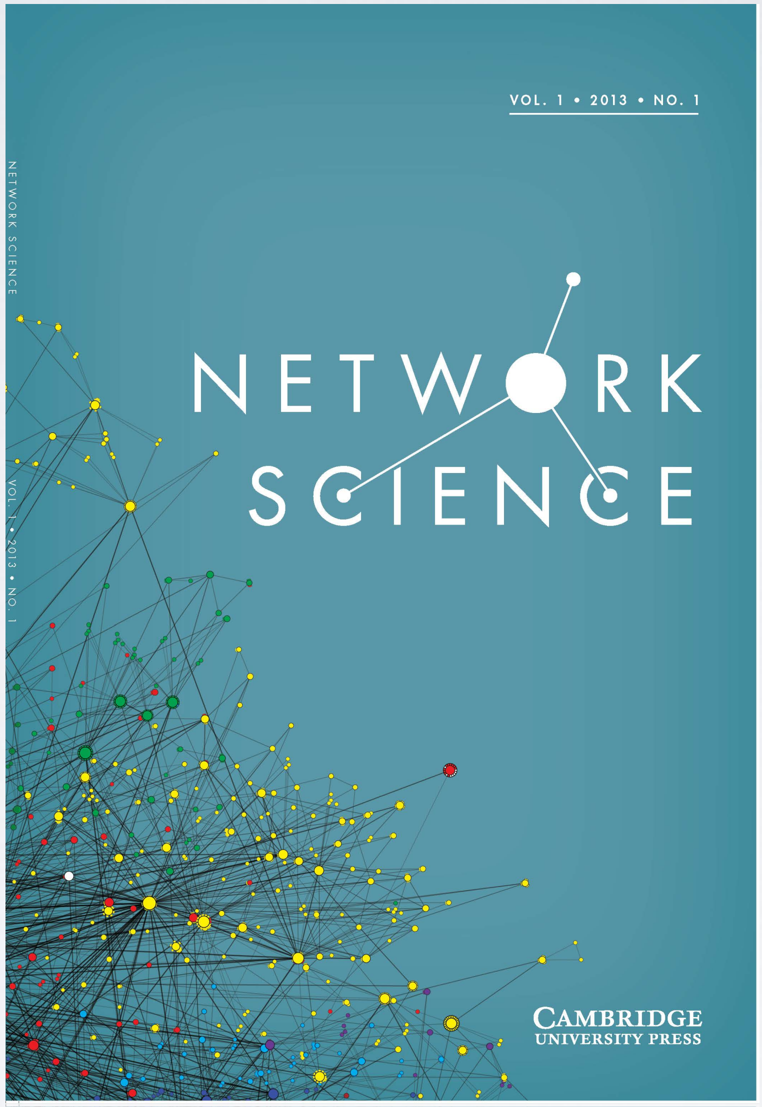
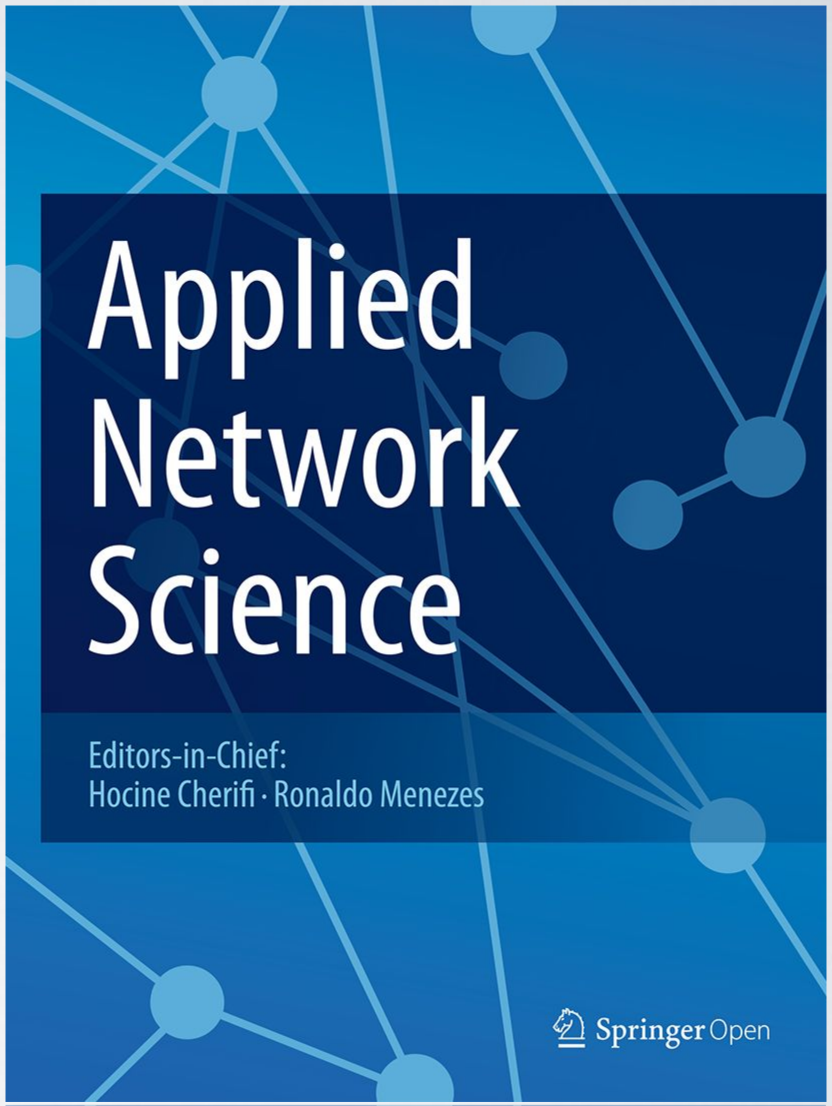
Materials

Journals







CONFERENCES

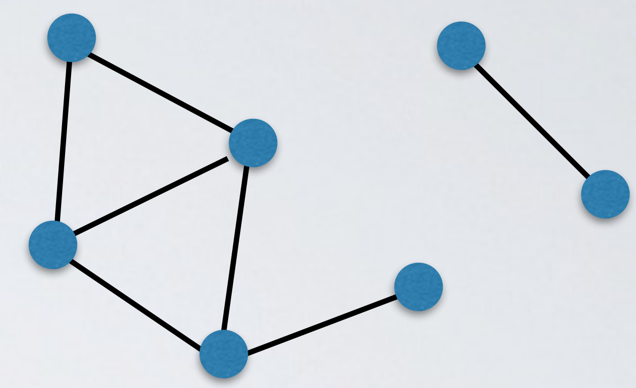
- NetSci, NetSci X - The Network Science Society (Since 2006)
- International Conference on Complex Networks and their Applications (Since 2011)
- CompleNet - International Conference on Complex Networks (Since 2009)
- France:
    - MARAMI (Modèles & Analyse des Réseaux : Approches Mathématiques & Informatiques) (Since 2009)

PROGRAM

Day	Time	Room	Type	Topic
Tuesday 09/15	10h15-12h15	F	Lecture	Introduction, Describing Networks
Thursday 09/17	10h15-12h15	F	Lecture	Centralities and Similarities
Thursday 09/24	08h-9h30	H	TD - Practicals	Python / Gephi / Network structure
Thursday 10/01	08h-10h	H	Lecture	Random Graph Models I: ER, Configuration, WS
Thursday 10/08	08h-10h	H	Lecture	Random Graph Models II: Scale Free, Barabasi, Forest Fire, etc.
Thursday 10/15	08h-10h	H	Lecture	Communities and Community Detection
Thursday 10/22	08h-10h	H	TD - Practicals	Project start
Thursday 11/05	10h15-12h15	?	Lecture	Dynamic Networks
Thursday 11/12	10h15-12h15	F	Lecture	Spreading Processes
Thursday 11/19	10h15-12h15	F	Lecture	*Guest Speakers: Complex Networks seen by researchers*
Thursday 11/26	10h15-12h15	F	Lecture	Complexifying Complex Networks (Multilayer, higher Order, Spatial, etc.)
Thursday 12/03	10h15-12h15	F	Lecture	Machine Learning on graphs (Link Prediction, Node classification, etc.)
Tuesday 12/08	10h15-12h15	F	TD - Practicals	Project / Python
Thursday 12/10	10h15-12h15	F	Lecture	Graph Embedding and Graph Convolutional Networks
Thursday 12/17	10h15-12h15	F	Lecture	Last class: article reading and commenting.

GRAPHS & NETWORKS

GRAPHS & NETWORKS



Networks often refers to real systems

- www,
- social network
- metabolic network.
- <u>Language: (Network, node, link)</u>

Vertex	Edge
person	friendship
neuron	synapse
Website	hyperlink
company	ownership
gene	regulation

Graph is the mathematical representation of a network

- <u>Language: (Graph, vertex, edge)</u>

In most cases we will use the two terms interchangeably.

GRAPH REPRESENTATION

NETWORK REPRESENTATIONS

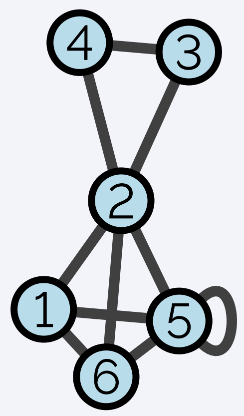
Networks: Graph notation

Graph notation : $G = (V, E)$

V	set of vertices/nodes.
E	set of edges/links.
$u \in V$	a node.
$(u, v) \in E$	an edge.

Network - Graph notation

Graph



Graph notation

$$G = (V, E)$$
$$V = \{1, 2, 3, 4, 5, 6\}$$
$$E = \{(1, 2), (1, 6), (1, 5), (2, 4), (2, 3), (2, 5), (2, 6), (6, 5), (5, 5), (4, 3)\}$$

NETWORK REPRESENTATIONS

- $G = (V, E)$
  - Often encoded as **edge list** or **adjacency list**
- Software: custom data structure and manipulation
  - add_nodes([i,j]), add_edge(i,j), …
- Libraries in many languages
  - Networkx (python)
  - igraph (python, C, R)
  - Graph-tools (python, C)

```
1 2
2 3
2 4
3 4
4 5
4 7
5 6
5 8
9 10
```

```
1    2
2    1 3 4
3    2 4
4    2 3 5 7
5    4 6 8
6    5
7    4
8    5
9    10
10   9
```

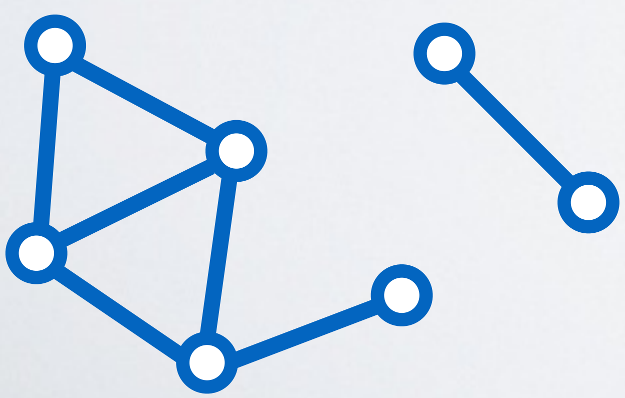
Types of Networks

Undirected networks

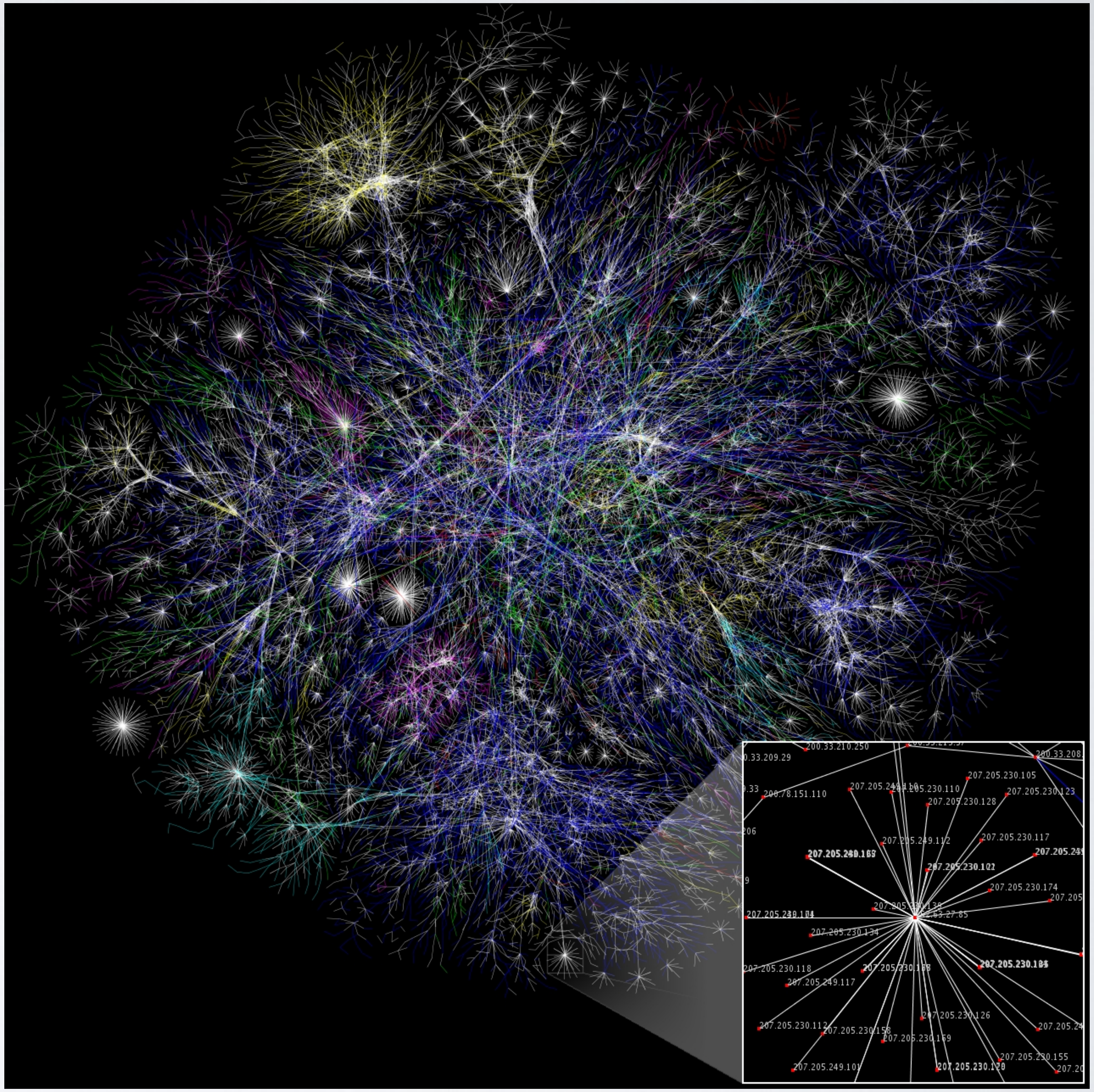
$G=(V, E)$

$(u,v) \in E \equiv (v,u) \in E$

- The directions of edges do not matter
- Interactions are possible between connected entities in both directions



Opte project



The Internet: Nodes - routers, Links - physical wires

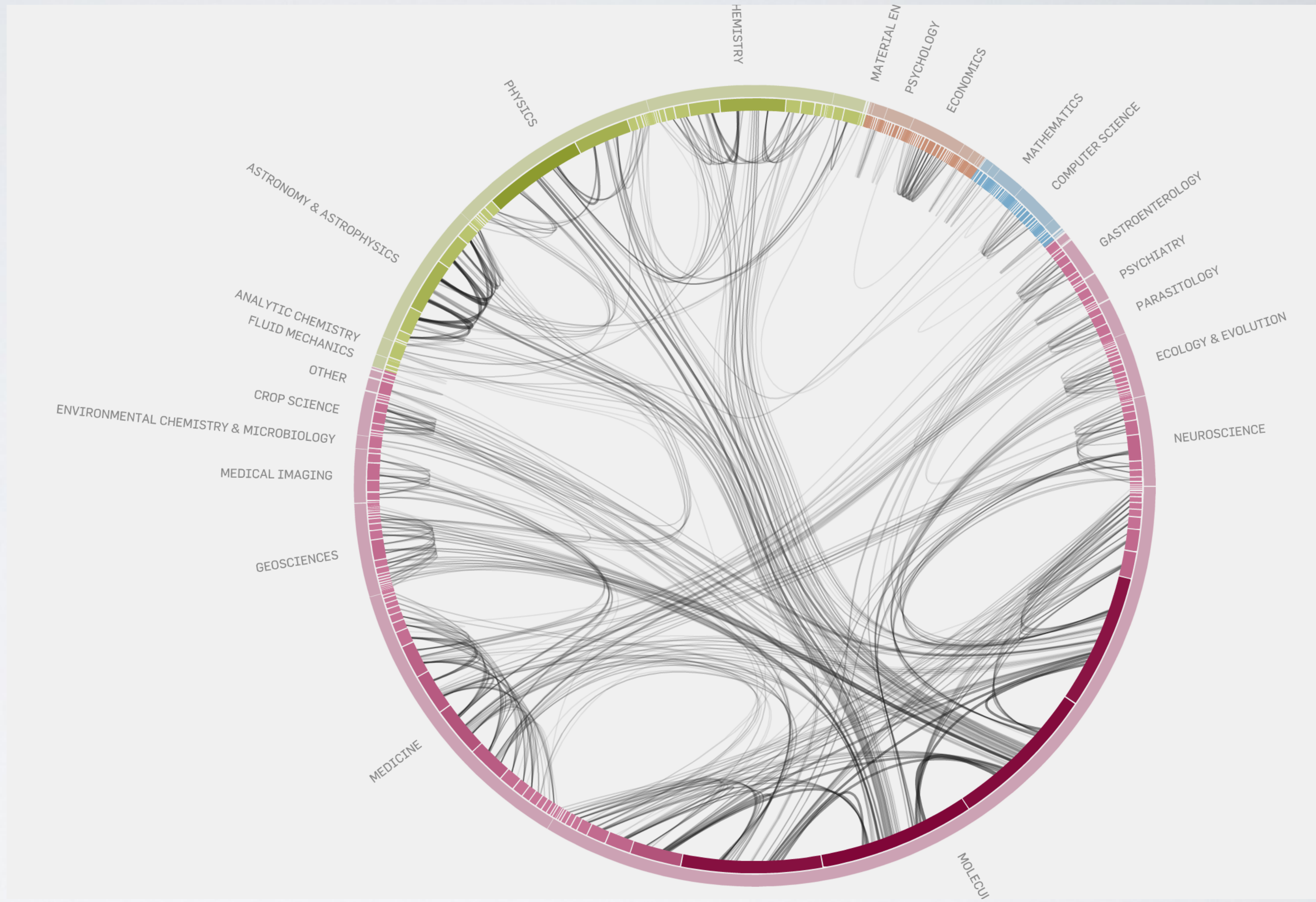
Directed networks

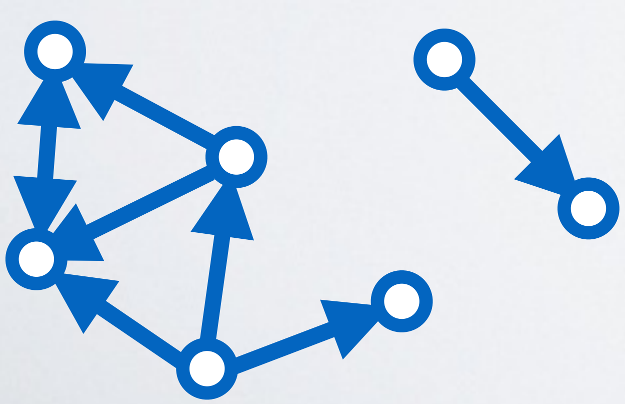
Moritz Stefaner, eigenfactor.com

$G=(V, E)$

$(u,v) \in E \not\equiv (v,u) \in E$

- The directions of edges matter
- Interactions are possible between connected entities only in specified directions

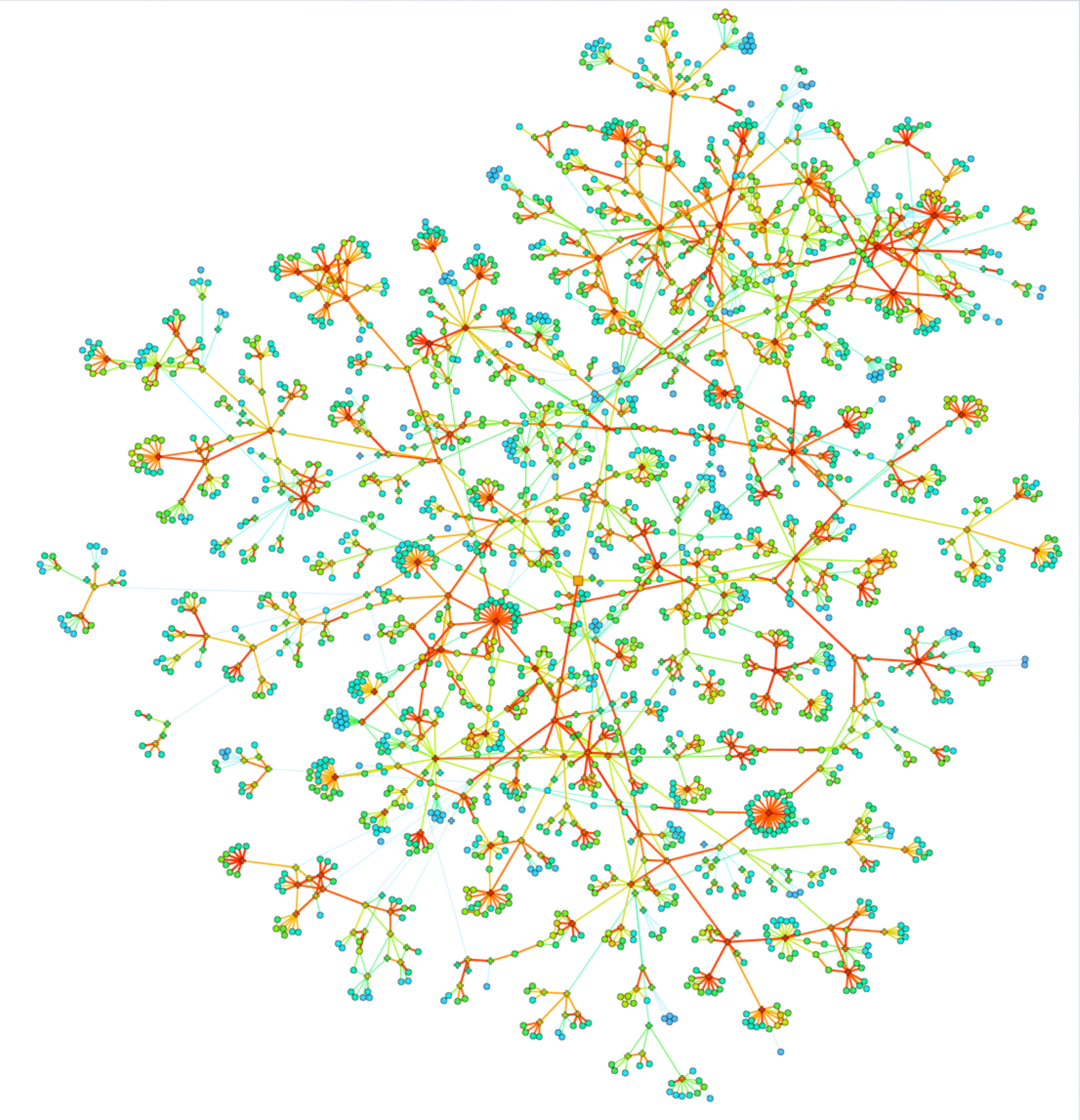




Citation network: Nodes - publications, Links - references

Weighted networks

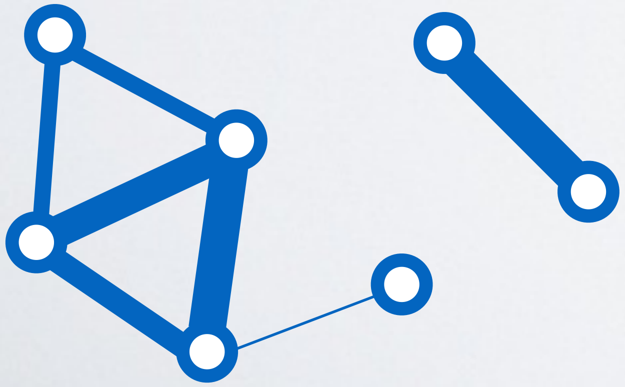
Onnela et.al. New Journal of Physics 9, 179 (2007).



$G=(V, E, w)$

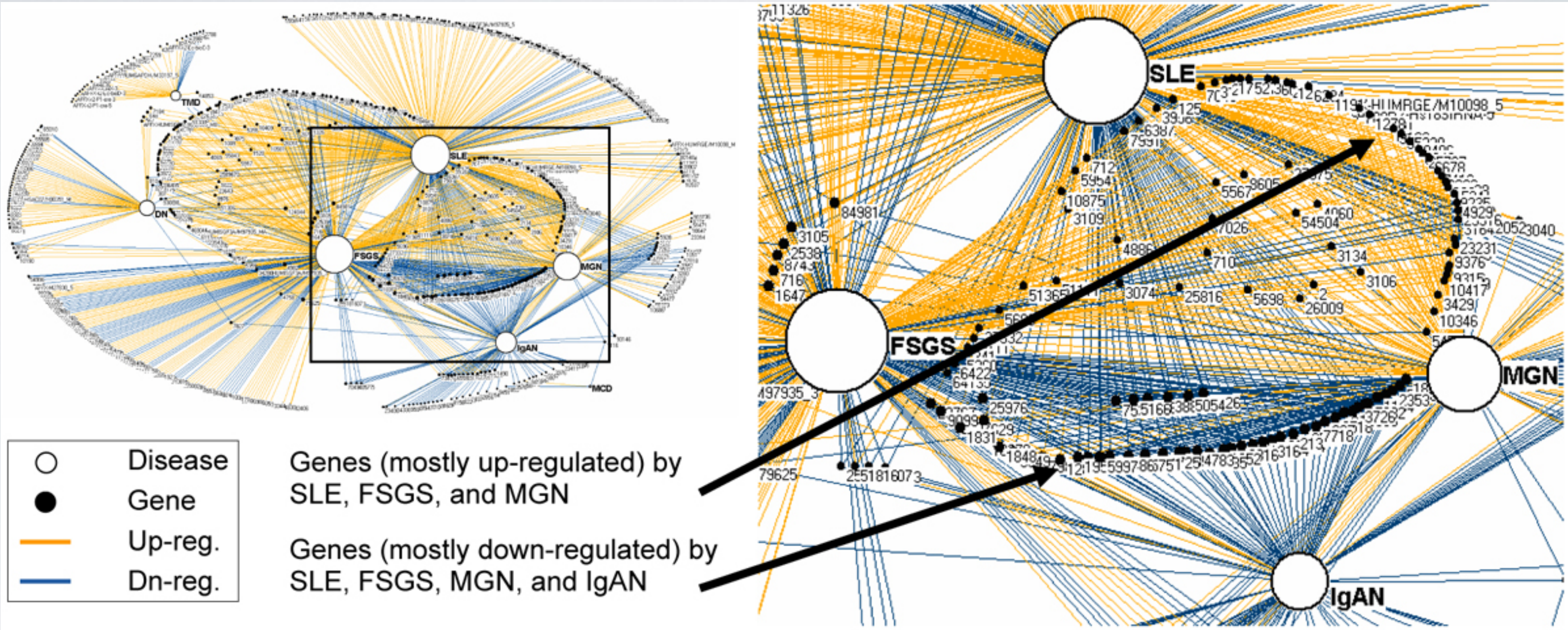
$w: (u,v) \in E \Rightarrow R$

- Strength of interactions are assigned by the weight of links

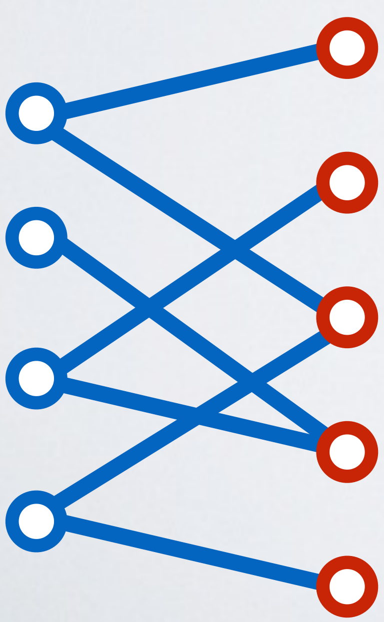


Social interaction network: Nodes - individuals
Links - social interactions

Bipartite network



Bhavnani et.al. BMC Bioinformatics 2009, **10**(Suppl 9):S3



Gene-desease network:

Nodes - Desease (7)&Genes (747)

Links - gene-desease relationship

$$G=(U, V, E)$$

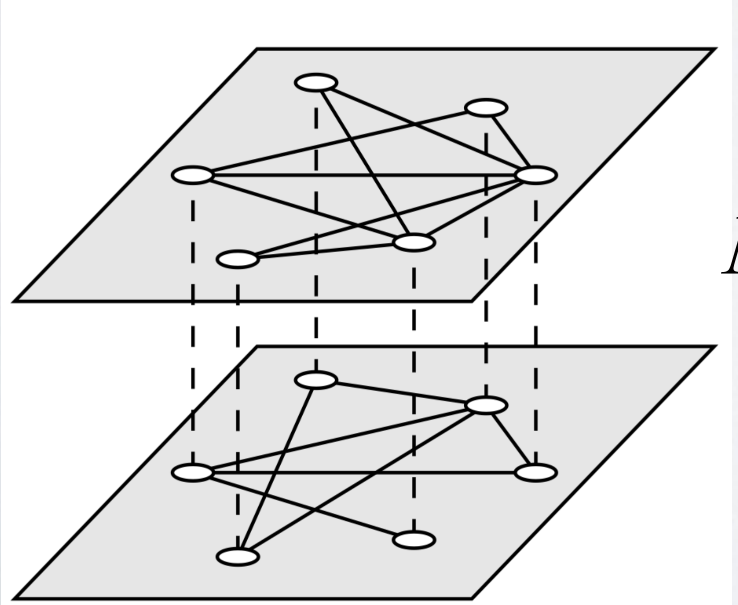
$$U \cap V = \emptyset$$

$$\forall (u,v) \in E,\ u \in U \ and\ v \in V$$

Multiplex and multilayer networks

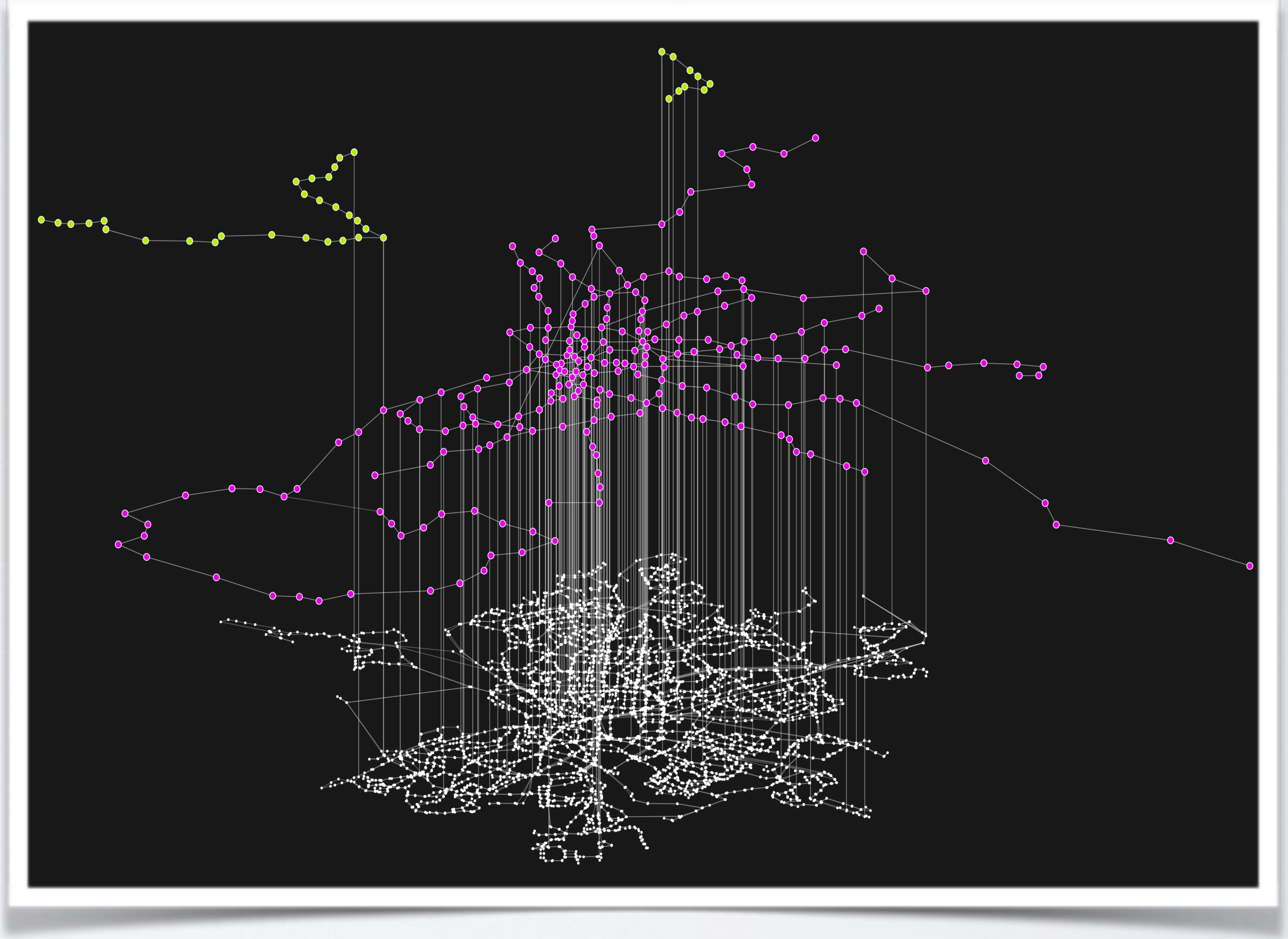
$G=(V, E_i),\ i=1\ldots M$

- Nodes can be present in multiple networks simultaneously
- These networks are connected (can influence each other) via the common nodes



Gomes et.al. Phys. Rev. Lett. 110, 028701 (2013)

$M=2$



[Mendez-Bermudez et al. 2017]

Temporal and evolving networks

$G=(V, E_t), (u,v,t,d) \in E_t$

t - time of interaction (u,v)
d - duration of interaction (u,v,t)

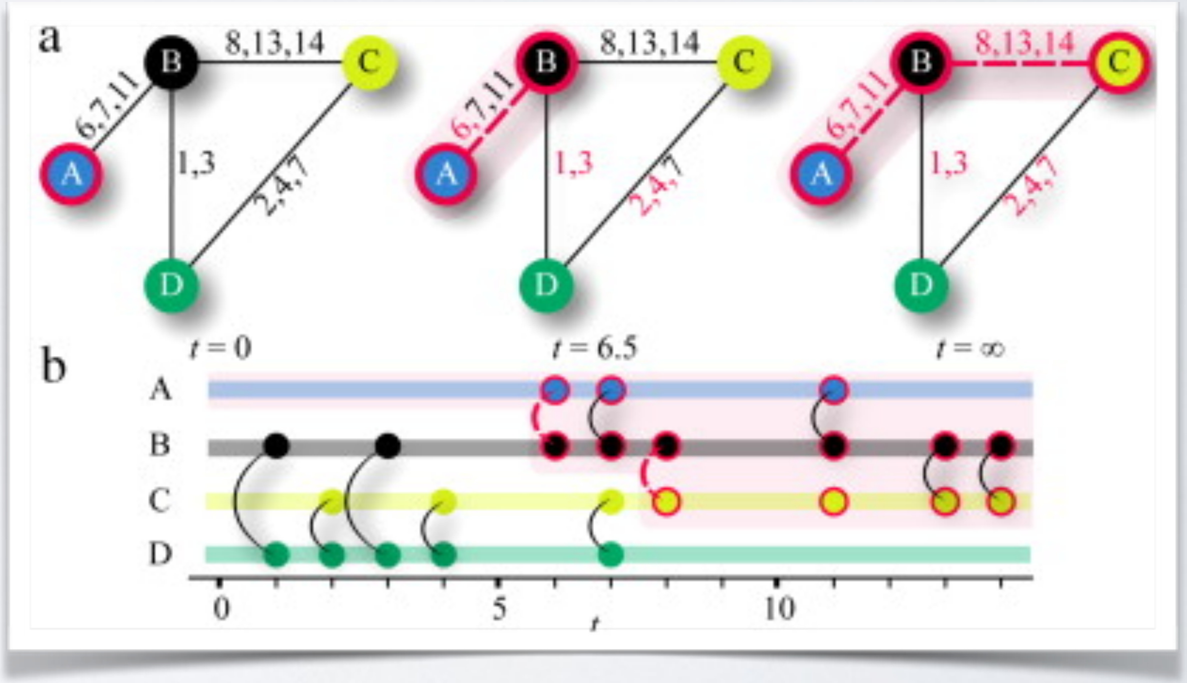
- Temporal links encode time varying interactions

$G=(V_{t'}, E_{t'})$

$v(t) \in V_{t'}$

$(u,v,t) \in E_{t'}$

- Dynamical nodes and links encode the evolution of the network



Mobile communication network
Nodes - individuals
Links - calls and SMS

NETWORK REPRESENTATIONS

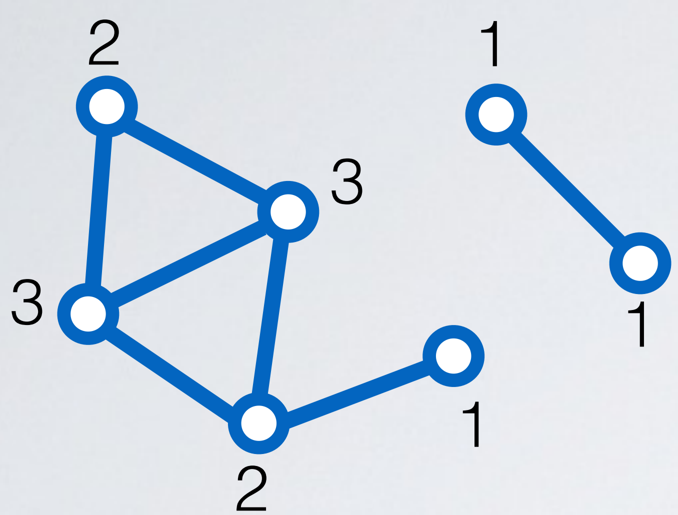
Node-Edge description

N_u	**Neighbourhood** of u, nodes sharing a link with u.
k_u	**Degree** of u, number of neighbors $\lvert N_u \rvert$.
N_u^{out}	**Successors** of u, nodes such as $(u, v) \in E$ in a directed graph
N_u^{in}	**Predecessors** of u, nodes such as $(v, u) \in E$ in a directed graph
k_u^{out}	**Out-degree** of u, number of outgoing edges $\lvert N_u^{out} \rvert$.
k_u^{in}	**In-degree** of u, number of incoming edges $\lvert N_u^{in} \rvert$
$w_{u,v}$	**Weight** of edge (u, v).
s_u	**Strength** of u, sum of weights of adjacent edges, $s_u = \sum_v w_{uv}$.

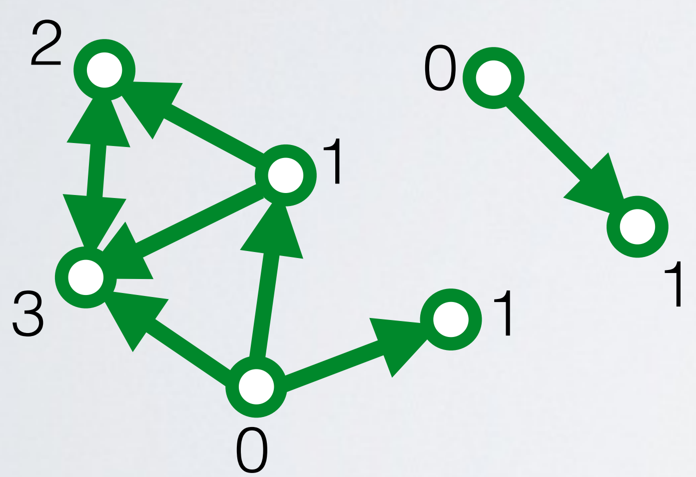
Node degree

Number of connections of a node

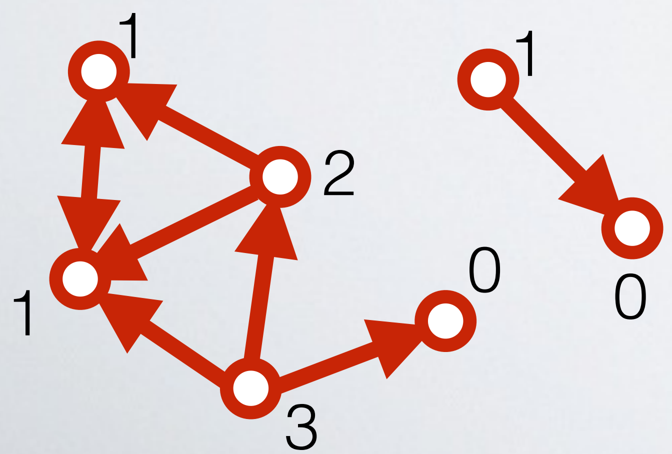
- Undirected network



- Directed network

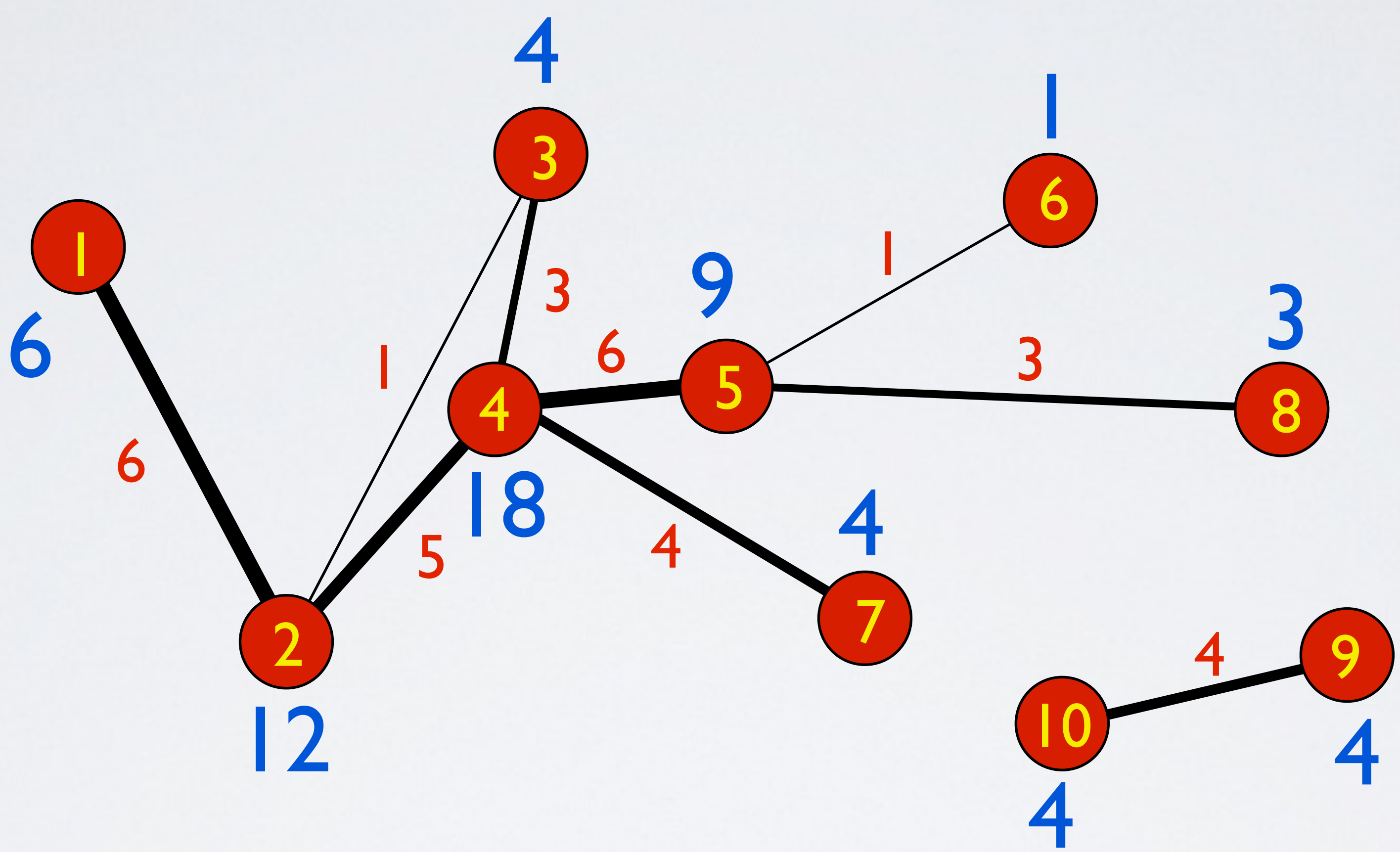


In degree



Out degree

Weighted degree: strength



DESCRIPTION OF GRAPHS

DESCRIPTION OF GRAPHS

- When confronted with a graph, how to describe it?
- How to compare graphs?
- What can we say about a graph?

SIZE

Counting nodes and edges

N/n	**size**: number of nodes $\|V\|$.
L/m	number of edges $\|E\|$
L_{max}	Maximum number of links

Undirected network: $\binom{N}{2} = N(N-1)/2$

Directed network: $\binom{N}{2} = N(N-1)$

SIZE

	#nodes (n)	#edges (m)
Wikipedia HL	2M	30M
Twitter 2015	288M	60B
Facebook 2015	1.4B	400B
Brain c. Elegans	280	6393
Roads US	2M	2.7M
Airport traffic	3k	31k

DENSITY

Network descriptors 1 - Nodes/Edges

$\langle k \rangle$	**Average degree**: Real networks are sparse, i.e., typically $\langle k \rangle \ll n$. Increases slowly with network size, e.g., $d \sim \log(m)$ $$\langle k \rangle = \frac{2m}{n}$$
$d/d(G)$	**Density**: Fraction of pairs of nodes connected by an edge in G. $$d = L/L_{\max}$$

DENSITY

	#nodes	#edges	Density	avg. deg
Wikipedia	2M	30M	1.5×10^{-5}	30
Twitter 2015	288M	60B	1.4×10^{-6}	416
Facebook	1.4B	400B	4×10^{-9}	570
Brain c.	280	6393	0,16	46
Roads Calif.	2M	2.7M	6×10^{-7}	2,7
Airport	3k	31k	0,007	21

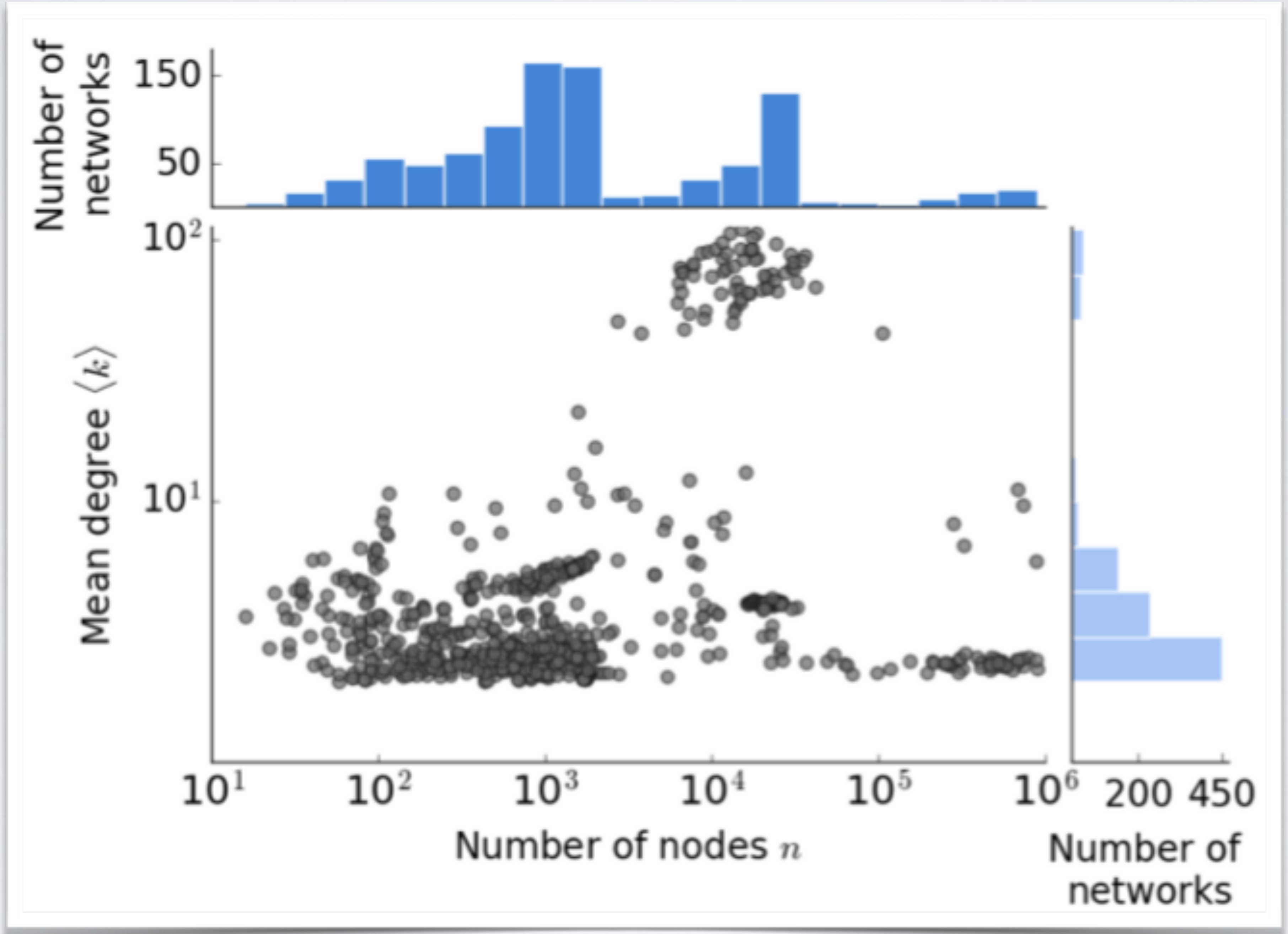
Beware: density hard to compare between graphs of different sizes

DENSITY

- It has been observed that: [Leskovec. 2006]
  - When graphs increase in size, the average degree increases
    - (Density on the contrary, decreases)
  - This increase is very slow
- Think of friends in a social network

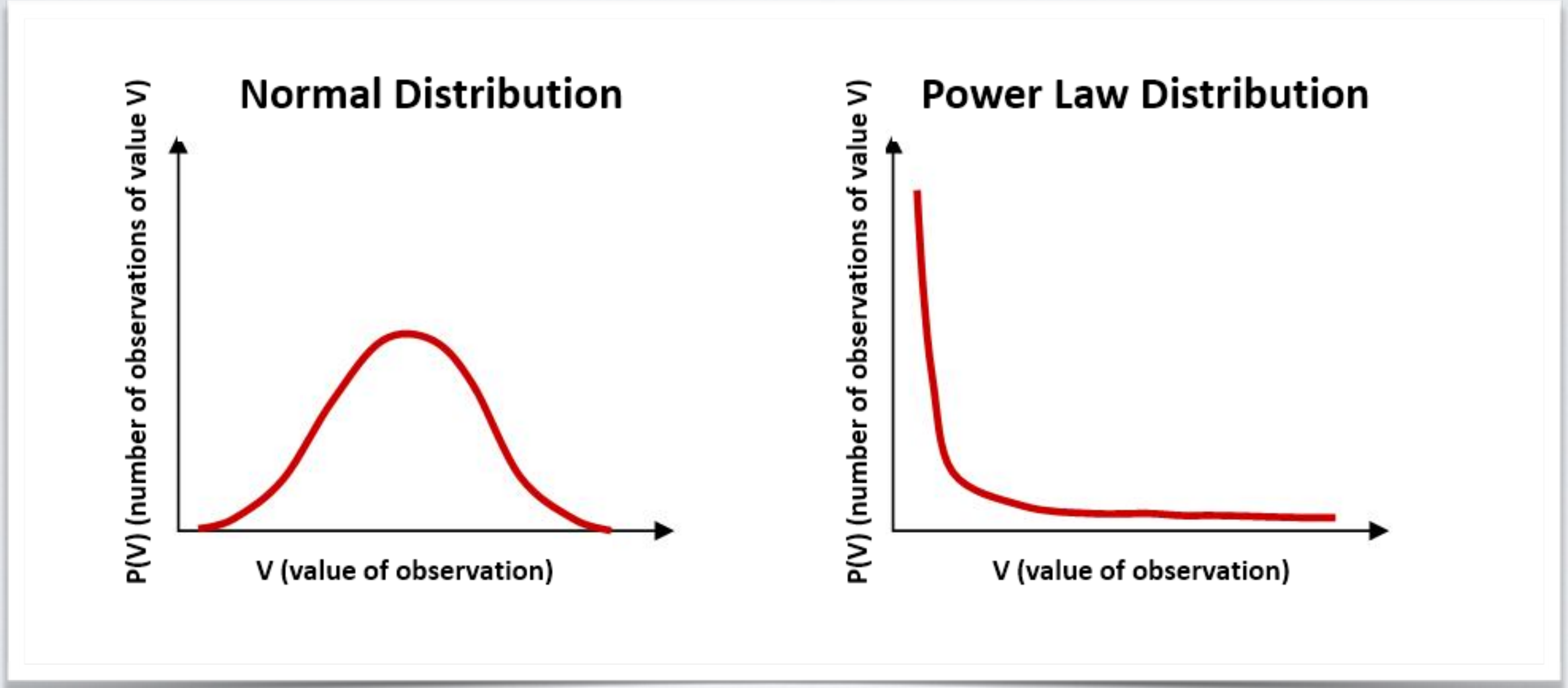
Leskovec, Jure, Jon Kleinberg, and Christos Faloutsos. "Graphs over time: densification laws, shrinking diameters and possible explanations." *Proceedings of the eleventh ACM SIGKDD international conference on Knowledge discovery in data mining*. 2005.

DENSITY



Broido, Anna D., and Aaron Clauset. "Scale-free networks are rare." *Nature communications* 10.1 (2019): 1-10.

DEGREE DISTRIBUTION



PDF (Probability Distribution Function)

DEGREE DISTRIBUTION

- In a fully random graph (Erdos-Renyi), degree distribution is (close to) a normal distribution centered on the average degree

- In real graphs, in general, it is not the case:
  - A high majority of small degree nodes
  - A small minority of nodes with very high degree (Hubs)

- Often modeled by a **power law**
  - More details later in the course

SUBGRAPHS

Subgraphs

subgraph $H(W)$: subset of nodes W of a graph $G = (V, E)$ and edges connecting them in G, i.e., subgraph $H(W) = (W, E'), W \subset V, (u, v) \in E' \iff u, v \in W \wedge (u, v) \in E$
Clique: subgraph with $d = 1$
Triangle: clique of size 3
Connected component: a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the supergraph
Strongly Connected component: In directed networks, a subgraph in which any two vertices are connected to each other by paths
Weakly Connected component: In directed networks, a subgraph in which any two vertices are connected to each other by paths if we disregard directions

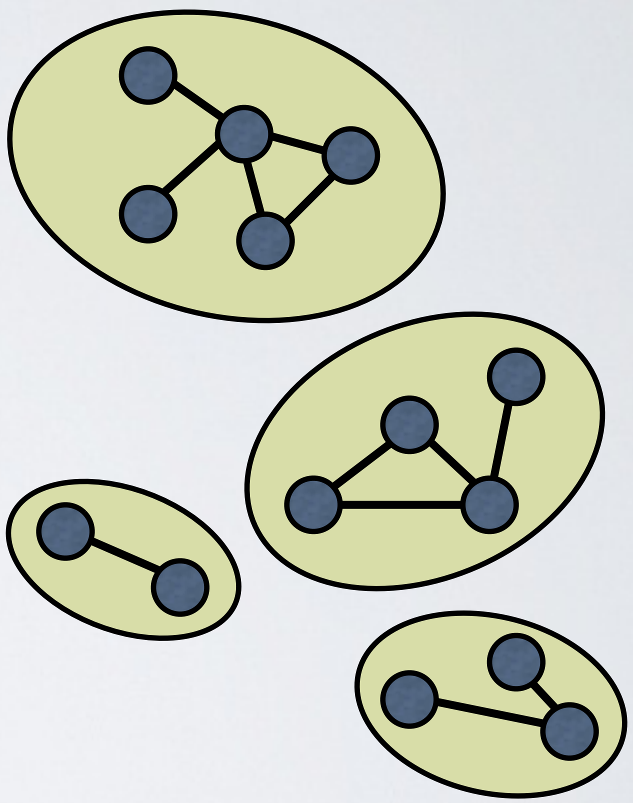


Figure after Newman, 2010

CLUSTERING COEFFICIENT

- **Clustering coefficient** or **triadic closure**
- Triangles are considered important in real networks
  - Think of social networks: *friends of friends are my friends*
  - # triangles is a big difference between real and random networks

CLUSTERING COEFFICIENT

Triangles counting

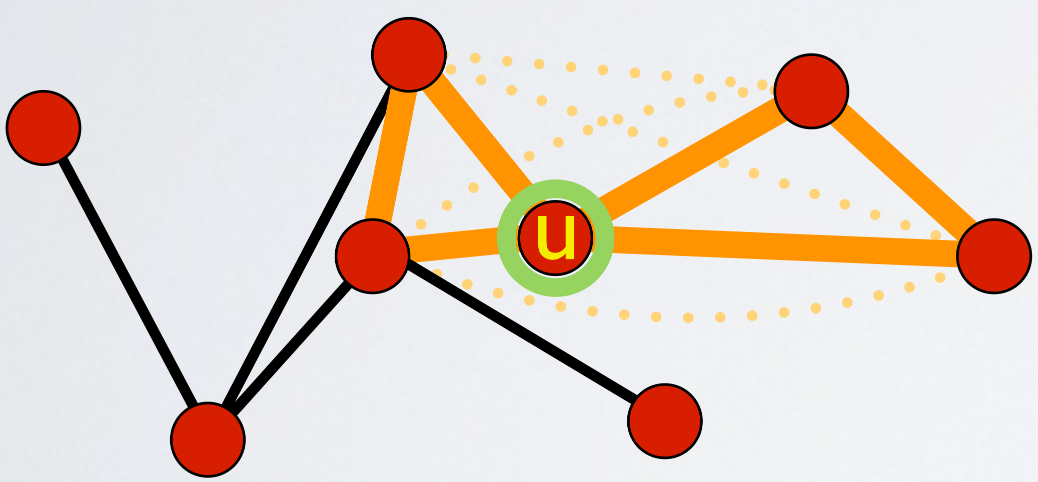
δ_u - **triads of u:** number of triangles containing node u
Δ - **number of triangles in the graph** total number of triangles in the graph, $\Delta = \frac{1}{3}\sum_{u \in V} \delta_u$.

Each triangle in the graph is counted as a triad once by each of its nodes.

$\delta_u^{\max}$ - **triads potential of u:** maximum number of triangles that could exist around node u, given its degree: $\delta_u^{\max} = \tau(u) = \binom{k_i}{2}$
$\Delta^{\max}$ - **triangles potential of G:** maximum number of triangles that could exist in the graph, given its degree distribution: $\Delta^{\max} = \frac{1}{3}\sum_{u \in V} \delta^{\max}(u)$

CLUSTERING COEFFICIENT

C_u - **Node clustering coefficient:** density of the subgraph induced by the neighborhood of u, $C_u = d(H(N_u))$. Also interpreted as the fraction of all possible triangles in N_u that exist, $\frac{\delta_u}{\delta_u^{\max}}$



Edges: 2

Max edges: 4*3/2=6

C_u=2/6=1/3

Triangles=2

Possible triangles= $\binom{4}{2}$ =6

C_u=2/6=1/3

CLUSTERING COEFFICIENT

$\langle C \rangle$ - **Average clustering coefficient:** Average clustering coefficient of all nodes in the graph, $\bar{C} = \frac{1}{N} \sum_{u \in V} C_u$.

Be careful when interpreting this value, since all nodes contributes equally, irrespectively of their degree, and that low degree nodes tend to be much more frequent than hubs, and their C value is very sensitive, i.e., for a node u of degree 2, $C_u \in 0, 1$, while nodes of higher degrees tend to have more contrasted scores.

C^g - **Global clustering coefficient:** Fraction of all possible triangles in the graph that do exist, $C^g = \frac{3\Delta}{\Delta^{\max}}$

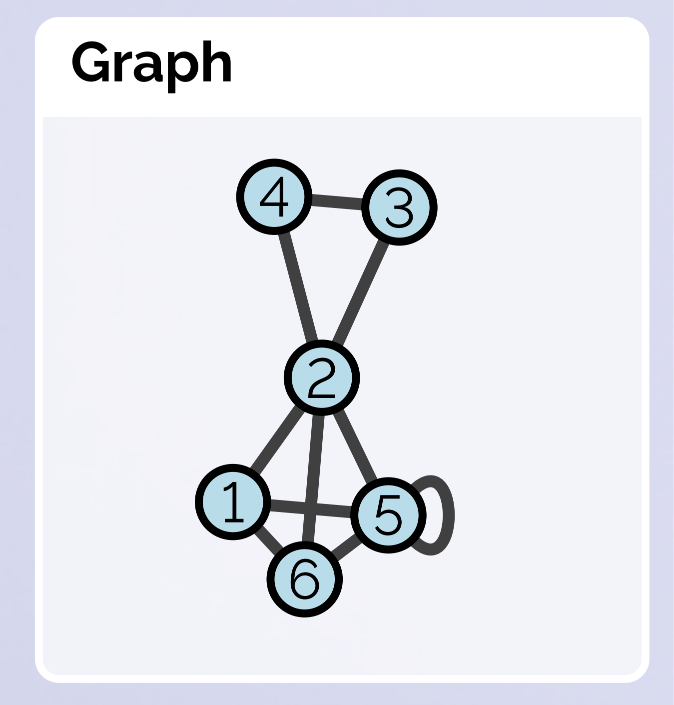
CLUSTERING COEFFICIENT

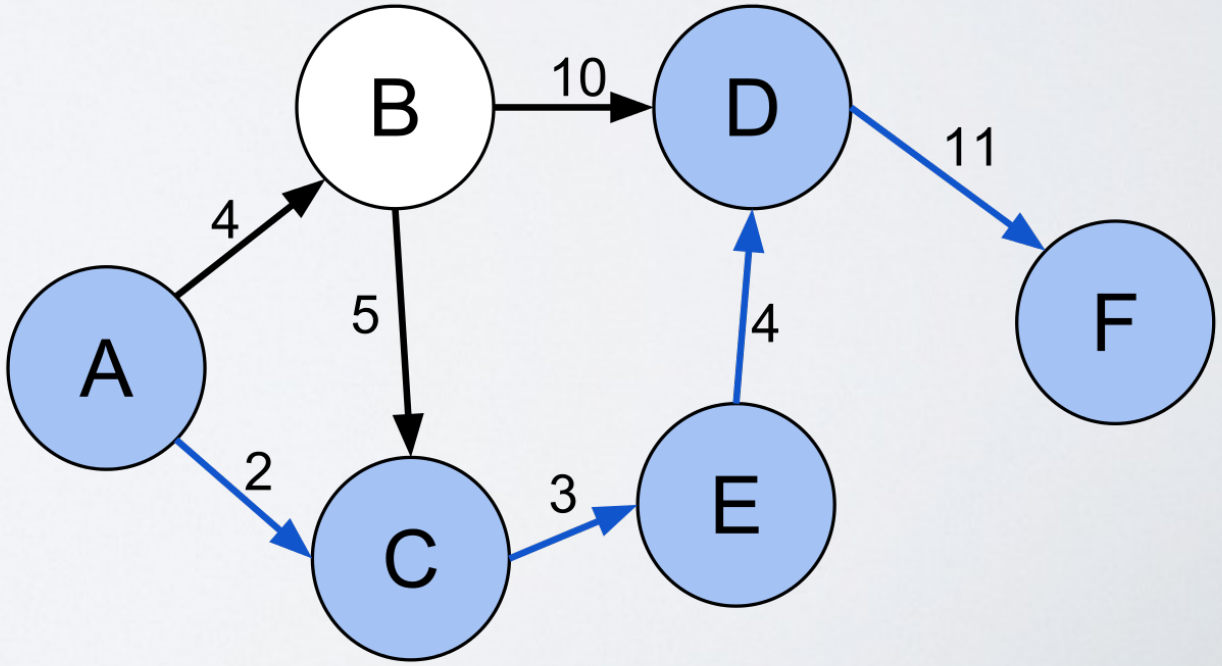
- Global CC:
  - In random networks, GCC = density
    - =>very small for large graphs
  - Facebook ego-networks: 0.6
  - Twitter lists: 0.56
  - California Road networks: 0.04

PATH RELATED SCORES

Paths - Walks - Distance

Walk: Sequences of adjacent edges or nodes (e.g., **1.2.1.6.5** is a valid walk)
Path: a walk in which each node is distinct.
Path length: number of edges encountered in a path
Weighted Path length: Sum of the weights of edges on a path
Shortest path: The shortest path between nodes u, v is a path of minimal *path length*. Often it is not unique.
Weighted Shortest path: path of minimal *weighted path length*.
$\ell_{u,v}$: **Distance**: The distance between nodes u, v is the length of the shortest path





All shortest path algorithm

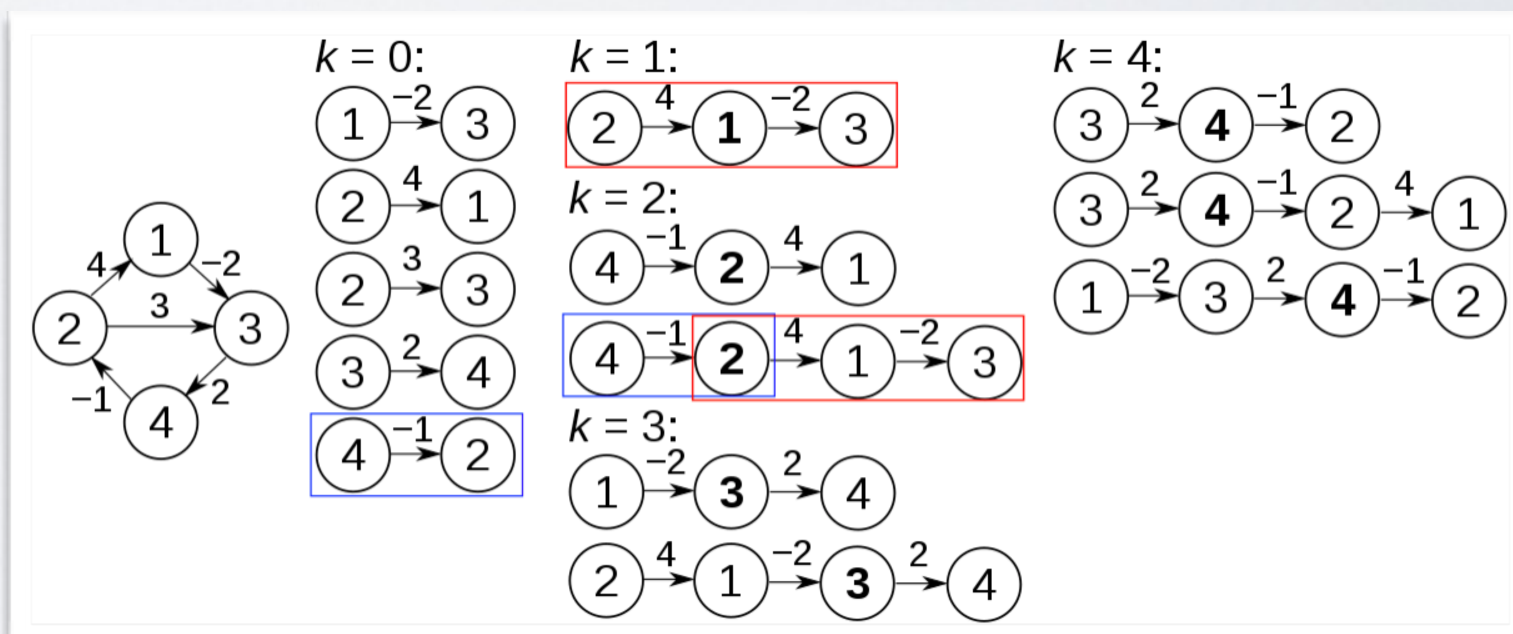
finding shortest paths in a **weighted graph** with **positive** or **negative edge weights** (but with no negative cycles)

```
proc FloydWarshall(G=(V,E,w))
// let dist be a |V| × |V| array of minimum distances initialized to ∞ (infinity)
for each edge (u,v)
   dist[u][v] ← w(u,v)  // the weight of the edge (u,v)
for each vertex v
   dist[v][v] ← 0
for k from 1 to |V|
   for i from 1 to |V|
      for j from 1 to |V|
         if dist[i][j] > dist[i][k] + dist[k][j]
             dist[i][j] ← dist[i][k] + dist[k][j]
         end if
```

Checking and updating all paths going through nodes k=1, 2, 3, … , N by assuming that:

shp(i,j,k)=
min(shp(i,j,k-1)), shp(i,k,k-1)+shp(k,j,k-1))

Complexity: $O(n^3)$



PATH RELATED SCORES

Network descriptors 2 - Paths

$\ell_{\max}$	**Diameter**: maximum *distance* between any pair of nodes.
$\langle \ell \rangle$	**Average distance**:

$$\langle \ell \rangle = \frac{1}{n(n-1)} \sum_{i \neq j} d_{ij}$$

AVERAGE PATH LENGTH

- The famous 6 degrees of separation (Milgram experiment)
  - (More on that next slide)
- Not too sensible to noise
- Tells you if the network is "stretched" or "hairball" like

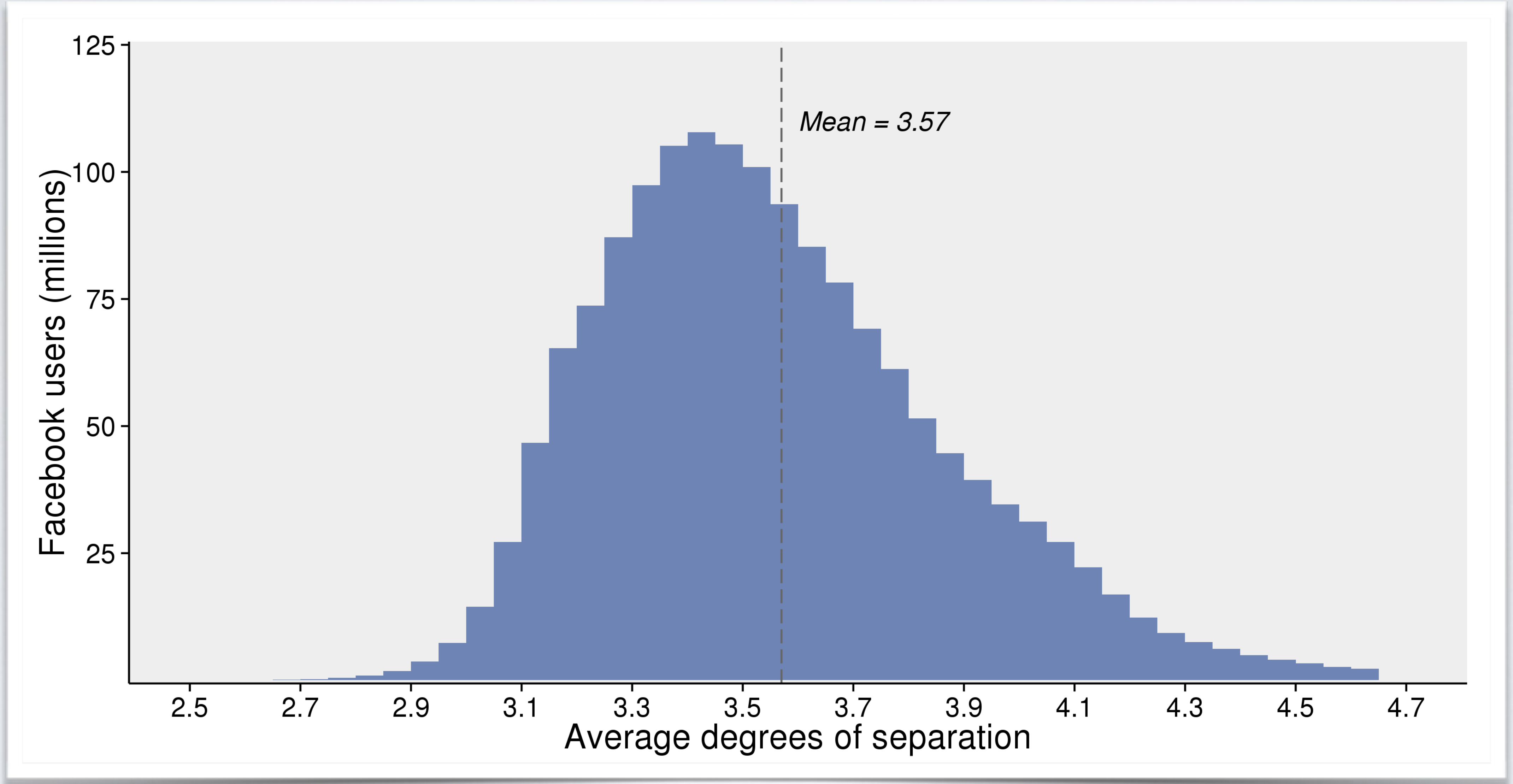
SIDE-STORY: MILGRAM EXPERIMENT

- Small world experiment (60's)
  - Give a (physical) mail to random people
  - Ask them to send to someone they don't know
    - They know his city, job
  - They send to their most relevant contact
- Results: In average, 6 hops to arrive

SIDE-STORY: MILGRAM EXPERIMENT

- Many criticism on the experiment itself:
  - Some mails did not arrive
  - Small sample
  - …
- Checked on “real” complete graphs (giant component):
  - MSN messenger
  - Facebook
  - The world wide web
  - …

SIDE-STORY: MILGRAM EXPERIMENT



Facebook

SMALL WORLD

Small World Network

A network is said to have the **small world** property when it has some structural properties. The notion is not quantitatively defined, but two properties are required:

- Average distance must be short, i.e., $\langle \ell \rangle \approx \log(N)$
- Clustering coefficient must be high, i.e., much larger than in a random network , e.g., $C^g \gg d$, with d the network density

More on this during the random network class

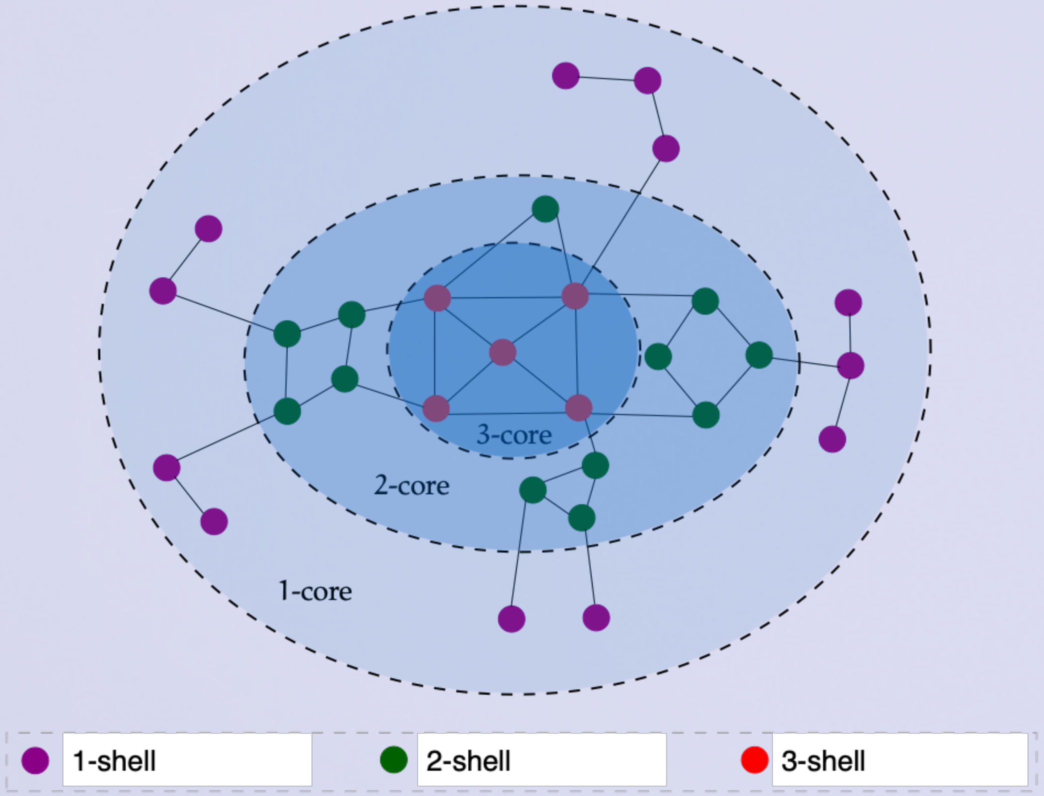
CORE-PERIPHERY : CORENESS

Goal: To identify dense cores of high degree nodes in networks

Cores and Shells

Many real networks are known to have a **core-periphery** structure, i.e., there is a densely connected core at its center and a more peripheral zone in which nodes are loosely connected between them and to the core.

k-core: The k-core (core of order k) of $G(V,E)$ is the largest subgraph $H(C)$ such as all nodes have at least a degree k, i.e., $\forall u \in C, k_u^H \leq k$, with k_u^H the degree of node u in subgraph H.
coreness: A vertex u has coreness k if it belongs to the k-core but not to the $k+1$-core.
c-shell: all vertices whose coreness is exactly c.



- A k-core of G can be obtained by recursively removing all the vertices of degree less than k, until all vertices in the remaining graph have at least degree k.

core decomposition

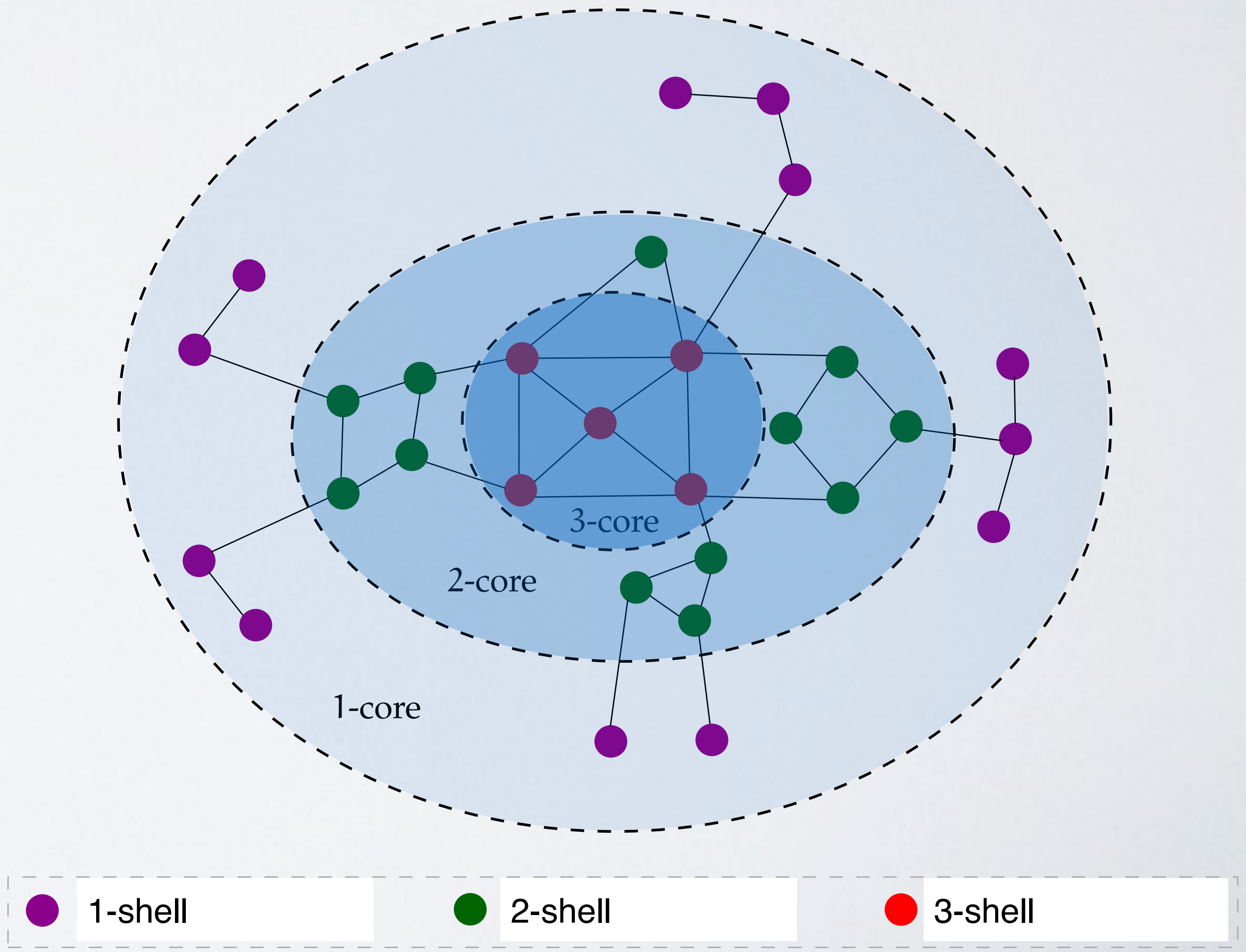
Intuitive algorithm

1. Take a directed or undirected network
2. Remove nodes with degree $k(=1)$ and all of those which degree became $k(=1)$ because of the removal process
3. Repeat step 2 for $k=2,3,\ldots$ until no node can be removed

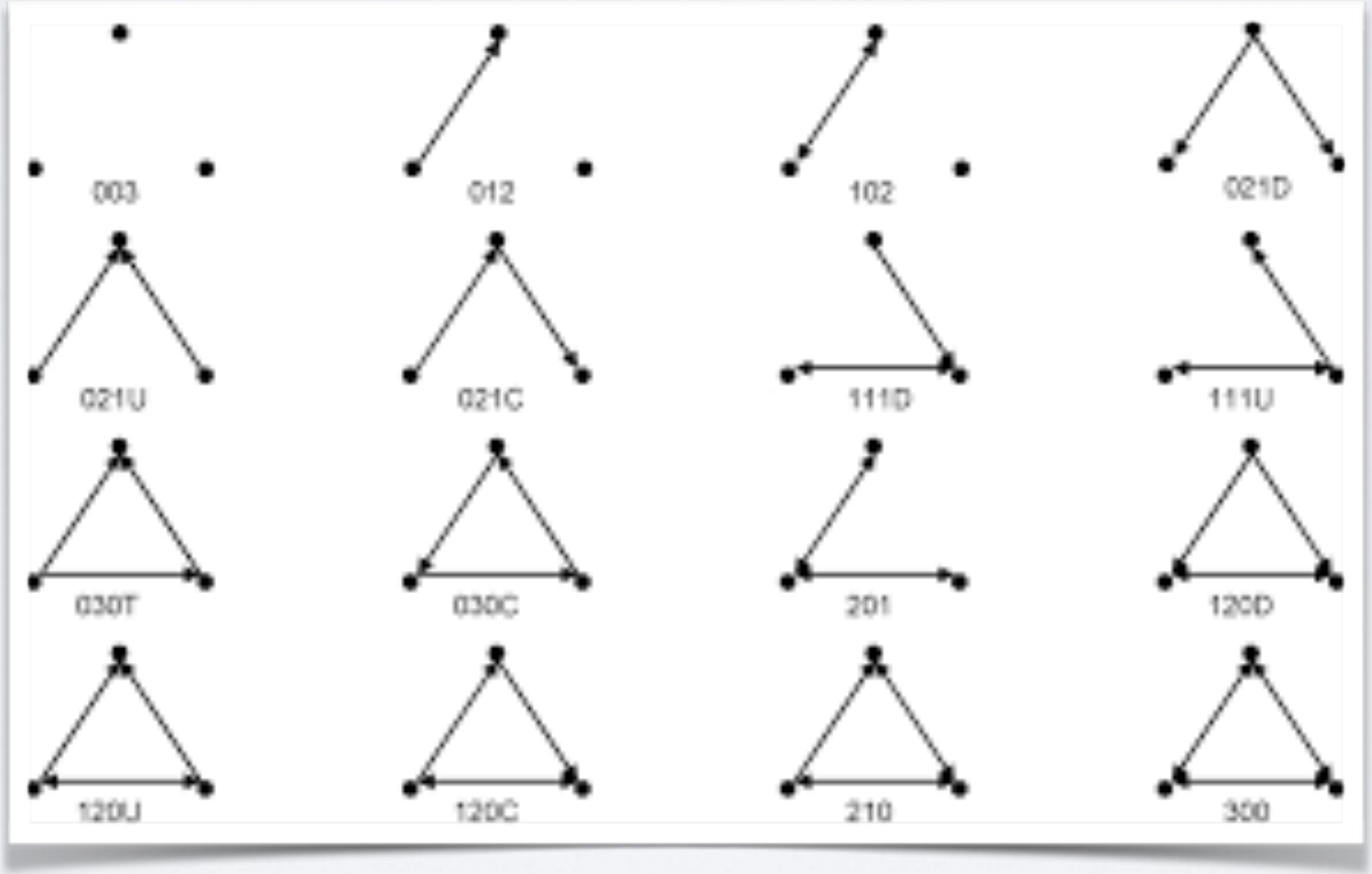
- Nodes removed in the kth turn are in the k-shell and the remaining nodes form the k-core

```
proc CoreDecomposition(G=(V,E))
   compute the degrees of vertices
   order v∈V in increasing degree order
   core[V]=0
   for each v∈V in the order
      core[v] := degree[v];
      for each u ∈ adj(v) do
         if deg[u] > deg[v] then
            degree[u] := degree[u] - 1;
            reorder V
         end if
```

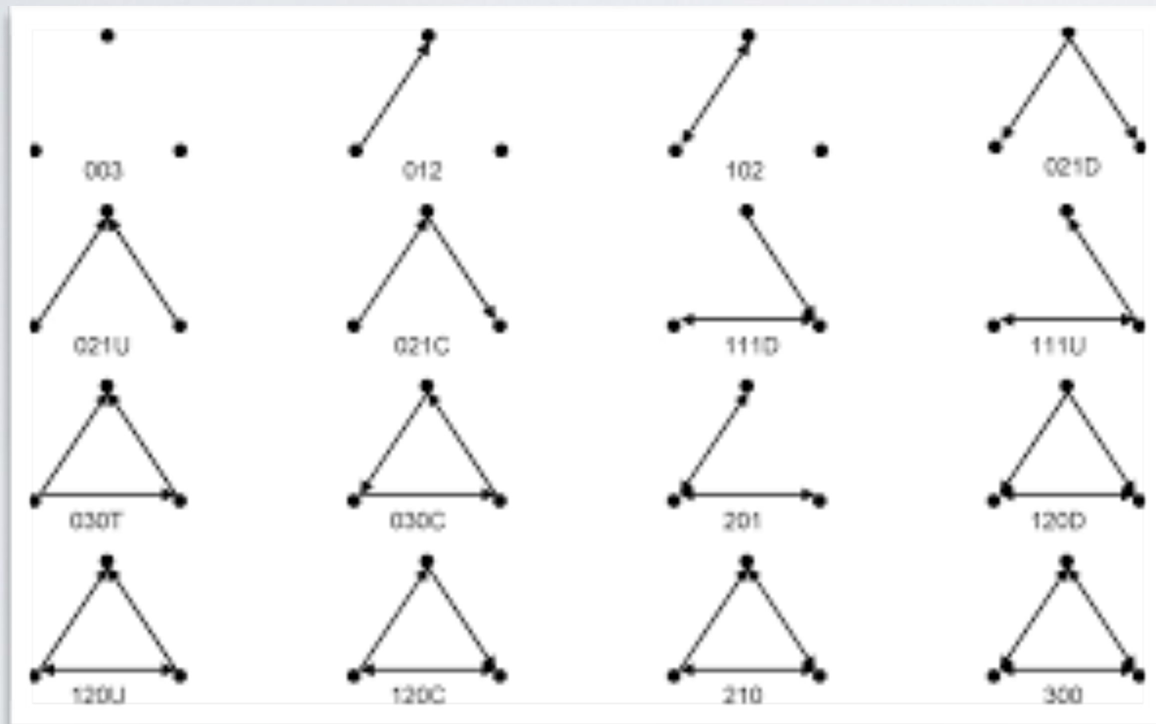
Batagelj, Zversnik (2002)

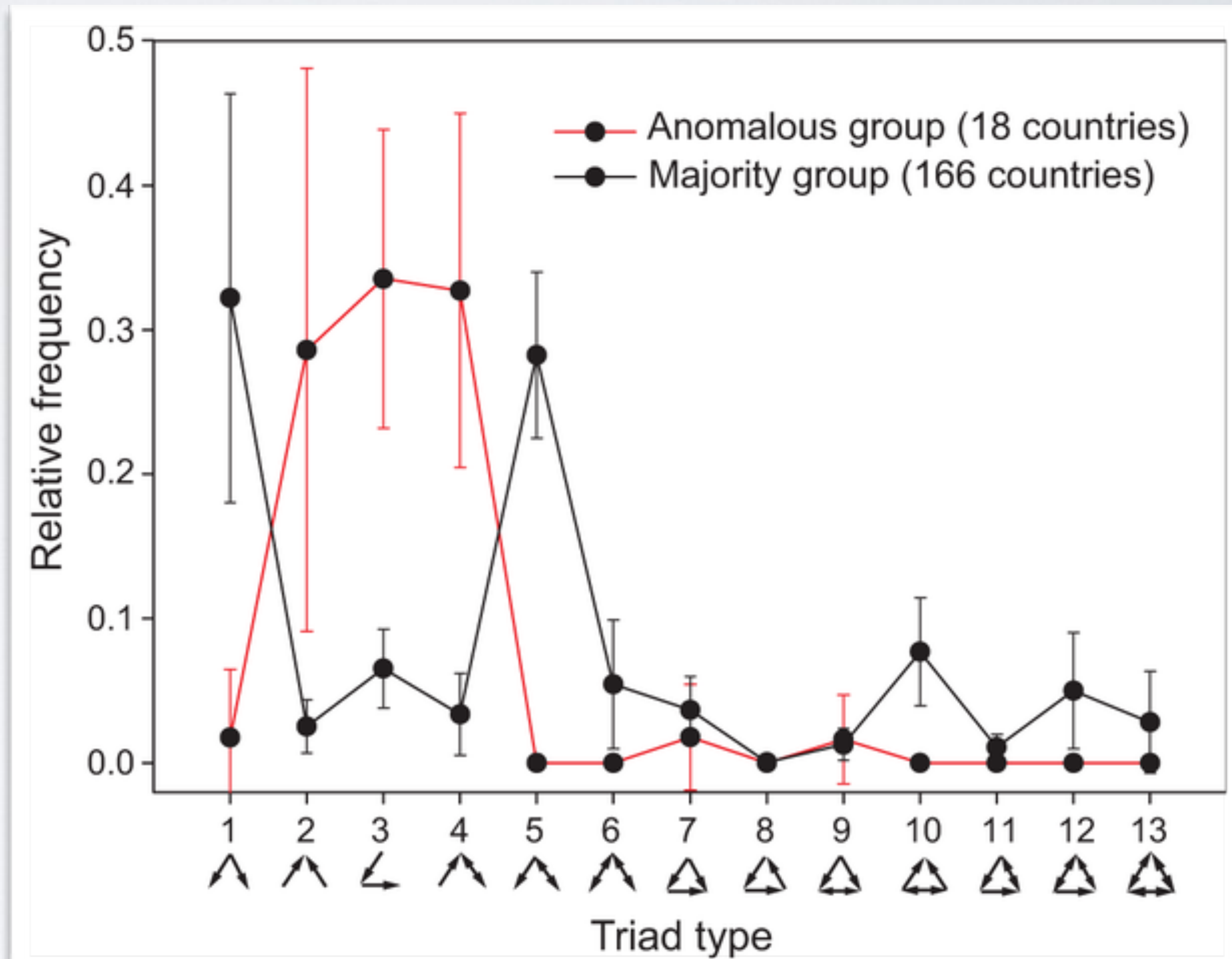


TRIADS COUNTING

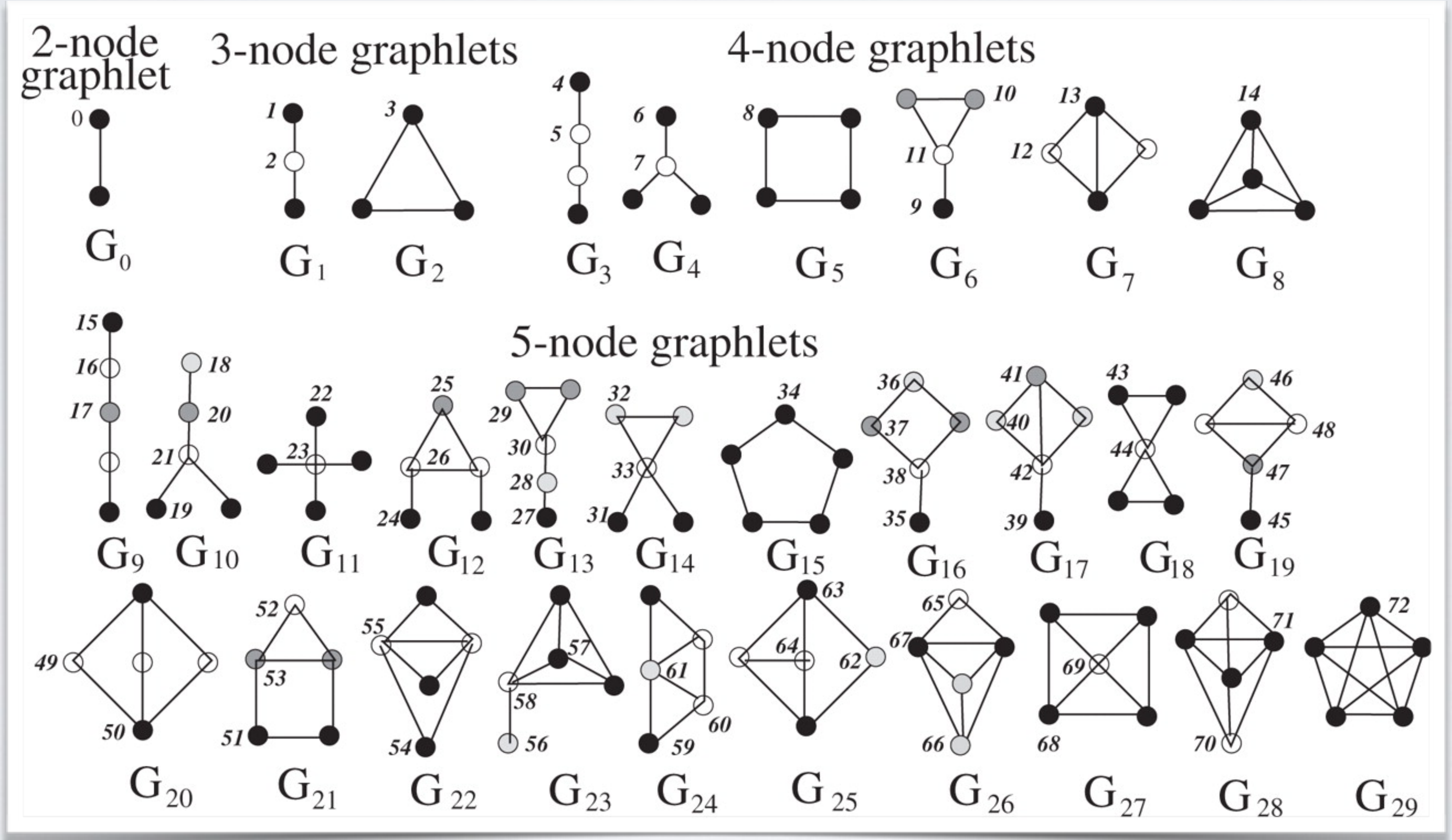


TRIADS COUNTING





GRAPHLETS


2-node graphlet
3-node graphlets
4-node graphlets
5-node graphlets
G0 G1 G2 G3 G4 G5 G6 G7 G8
G9 G10 G11 G12 G13 G14 G15 G16 G17 G18 G19
G20 G21 G22 G23 G24 G25 G26 G27 G28 G29

GRAPHS AS MATRICES

Matrices in short

Matrices are mathematical objects that can be thought as *tables* of numbers. The size of a matrix is expressed as $m \times n$, for a matrix with m rows and n columns. **The order (row/column) is important**.
M_{ij} is a notation representing the element on **row** m and **column** j.

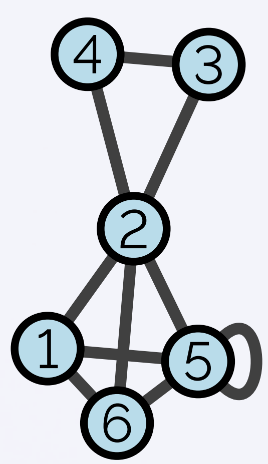
ADJACENCY MATRIX

A - Adjacency matrix

The most natural way to represent a graph as a matrix is called the Adjacency matrix A. It is defined as a square matrix, such as the number of rows (and the number of columns) is equal to the number of nodes N in the graph. Nodes of the graph are numbered from 1 to N, and there is an edge between nodes i and j if the corresponding position of the matrix A_{ij} is not 0.

- A value on the diagonal means that the corresponding node has a **self-loop**
- the graph is **undirected**, the matrix is **symmetric**: $A_{ij} = A_{ji}$ for any i, j.
- In an **unweighted** network, and edge is represented by the value 1.
- In a **weighted** network, the value A_{ij} represents the **weight** of the edge (i, j)

Graph



A - Adjacency Mat.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

ADJACENCY MATRIX

Typical operations on A

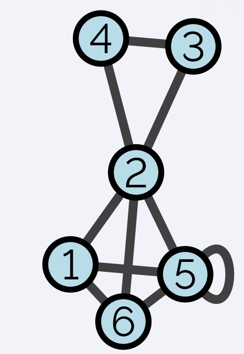
Some operations on Adjacency matrices have straightforward interpretations and are frequently used

---

Multiplying A by **itself** allows to know the number of walks of a given length that exist between any pair of nodes: A^2_{ij} corresponds to the number of walks of length 2 from node i to node j, A^3_{ij} to the number of walks of length 3, etc.

Multiplying A by a **column vector** W of length $1 \times N$ can be thought as setting the i th value of the vector to the ith node, and each node *sending* its value to its neighbors (for undirected graphs). The result is a column vector with N elements, the ith element corresponding to the sum of the values of its neighbors in W. This is convenient when working with **random walks** or **diffusion** phenomenon.

Graph



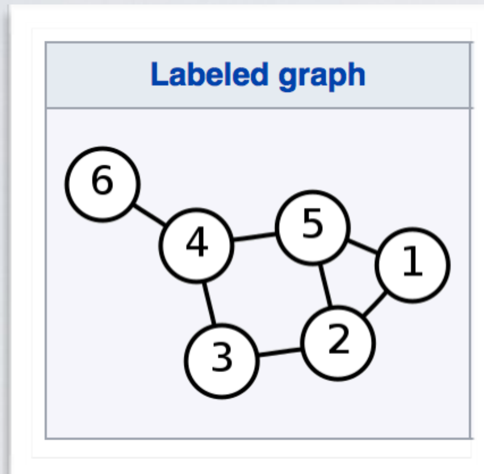
A - Adjacency Mat.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A^2

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 3 & 2 \\ 2 & 5 & 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 3 & 3 & 1 & 1 & 4 & 3 \\ 2 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}$$

ADJACENCY MATRIX



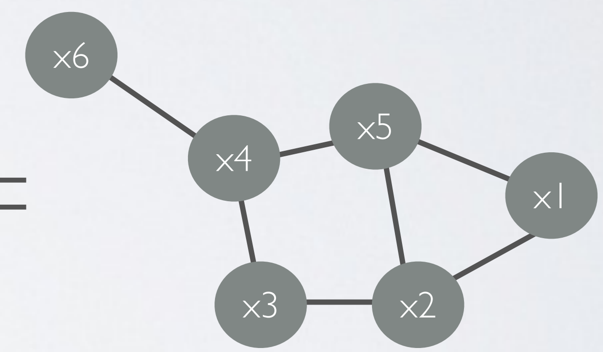
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_5 \\ x_5 \\ x_6 \end{pmatrix}$$

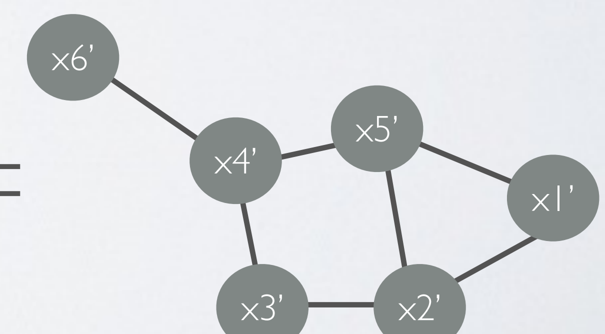
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_5 \\ x_5 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' = x_2 + x_4 \\ x_2' = x_1 + x_3 + x_5 \\ x_3' = x_2 + x_4 \\ x_4' = x_3 + x_5 + x_6 \\ x_5' = x_1 + x_2 + x_4 \\ x_6' = x_4 \end{pmatrix}$$

A, x =



A, Ax=



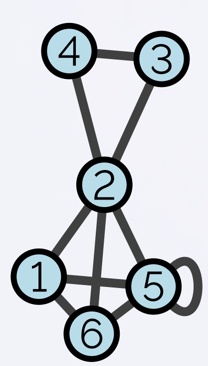
LAPLACIAN

Graph Laplacian

The **Graph Laplacian**, or **Laplacian Matrix** of a graph is a variant of the Adjacency matrix, often used in *Graph theory* and *Spectral Graph Theory*.
It is defined as $D - A$, with D the *Degree matrix* of the graph, defined as a $N \times N$ matrix with $D_{ii} = k_i$ and zeros everywhere else.

Intuitively, Laplace operator is a generalization of the second derivative, and is defined in discrete situations, for each value, as the sum of differences between the value and its "neighbors". e.g., in time, the 2$^{\text{nd}}$ derivative *acceleration* is the difference between current speed and previous speed. In a B&W picture, it's the difference between the greylevel on current pixel and the greylevel of 4 or 8 closest pixels, and perform *edge detection*. On a graph, with W a column vector representing values on nodes, LW computes for each node the difference to neighbors.

Graph



A - Adjacency Mat.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

D - Degree Matrix

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

L - Laplacian

$$\begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 4 & -1 \\ -1 & -1 & 0 & 0 & -1 & 3 \end{pmatrix}$$

SPECTRAL GRAPH THEORY

Spectral properties of A

Spectral Graph Theory is a whole field in itself, and beyond the scope of this class. A few elements for those with a *linear algebra* background:

- The adjacency matrix of an undirected simple graph is symmetric, and therefore has a complete set of real eigenvalues and an orthogonal eigenvector basis.
- The set of eigenvalues of a graph is the spectrum of the graph.
- Eigenvalues are denoted as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$
- The largest eigenvalue λ_0 lies between the average and maximum degrees
- The number of closed walks of length k in G equals $\sum_i^n = 0\lambda_i^k$
- A graph is bipartite if and only if its spectrum is symmetric (i.e., if λ is an eigenvalue, then so is $-\lambda$
- If G is connected, then the diameter of G is strictly less than its number of distinct eigenvalues

SPECTRAL GRAPH THEORY

Spectral properties of L

Eigenvalues of the Laplacian have many applications, such as *spectral clsutering*, *graph matching*, *embedding*, etc. Assuming G undirected with eigenvalues $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$, here are some interesting properties:

- The smallest eigenvalue λ_i equals 0
- The number of 0 eigenvalues gives the number of connected components

SPECTRAL GRAPH THEORY

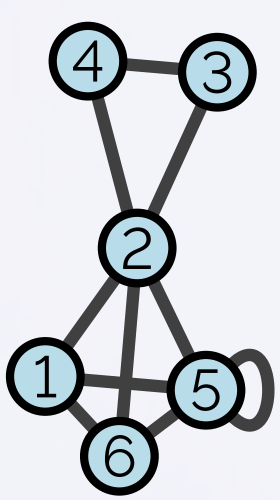
- Graph Spectral Analysis is a whole field of research
- We will introduce more of it in later parts of the course
  - Centralities
  - Community Detection
  - Embedding
  - …

RANDOM WALK MATRIX

Random Walk matrix

Another useful matrix of a graph is the **Random Walk Transition Matrix** R. It is the column normalized version of the adjacency matrix. R_{ij} can be understood as the probability for a random walker located on node i to move to j.

Graph



A - Adjacency Mat.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Random W. mat.

$$\begin{pmatrix} 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{5} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{5} & 0 & 0 & \frac{1}{4} & 0 \end{pmatrix}$$

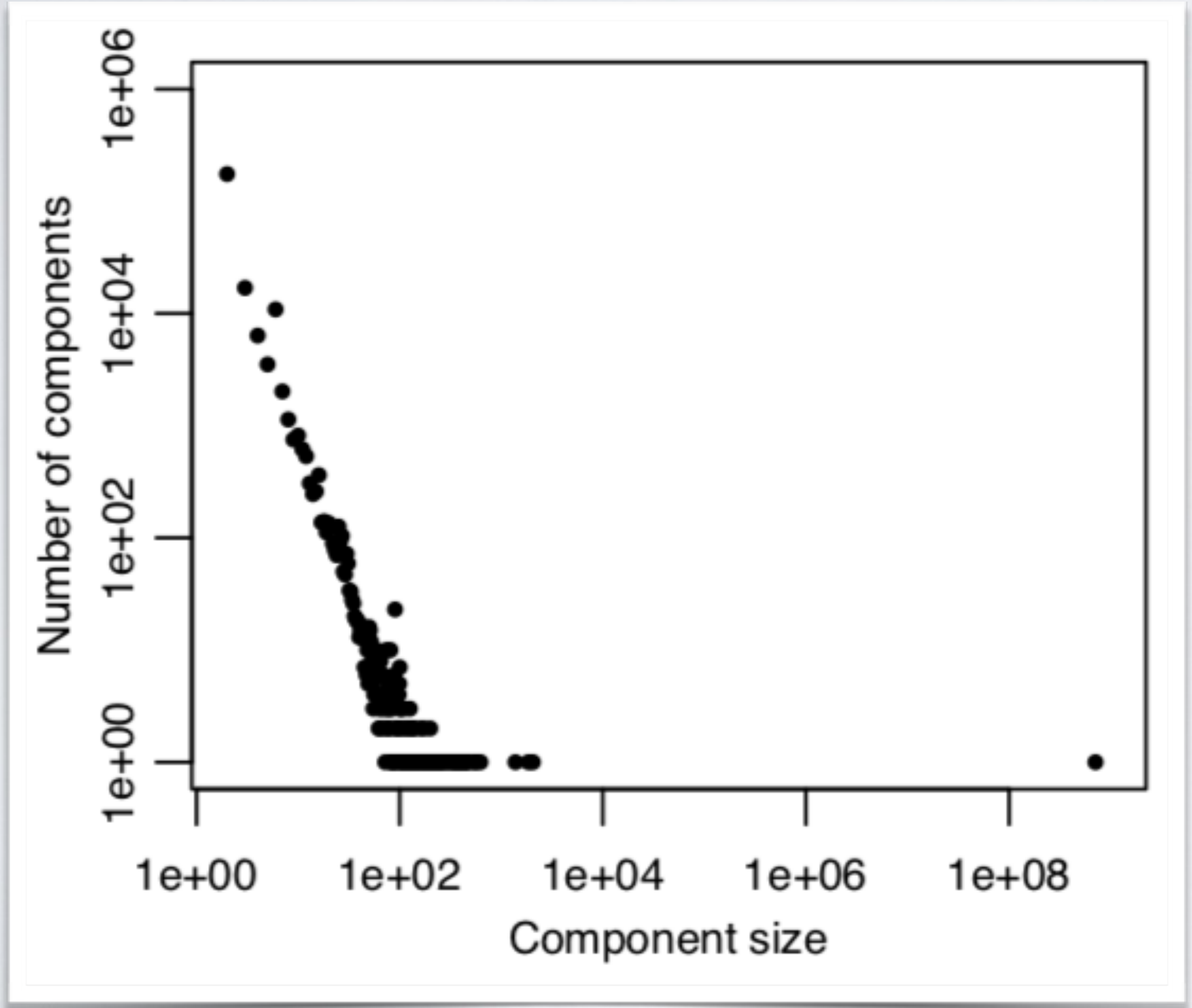
EXEMPLE OF GRAPH ANALYSIS

- Source: [The Anatomy of the Facebook Social Graph, Ugander et al. 2011]
- The Facebook friendship network in 2011

EXEMPLE OF GRAPH ANALYSIS

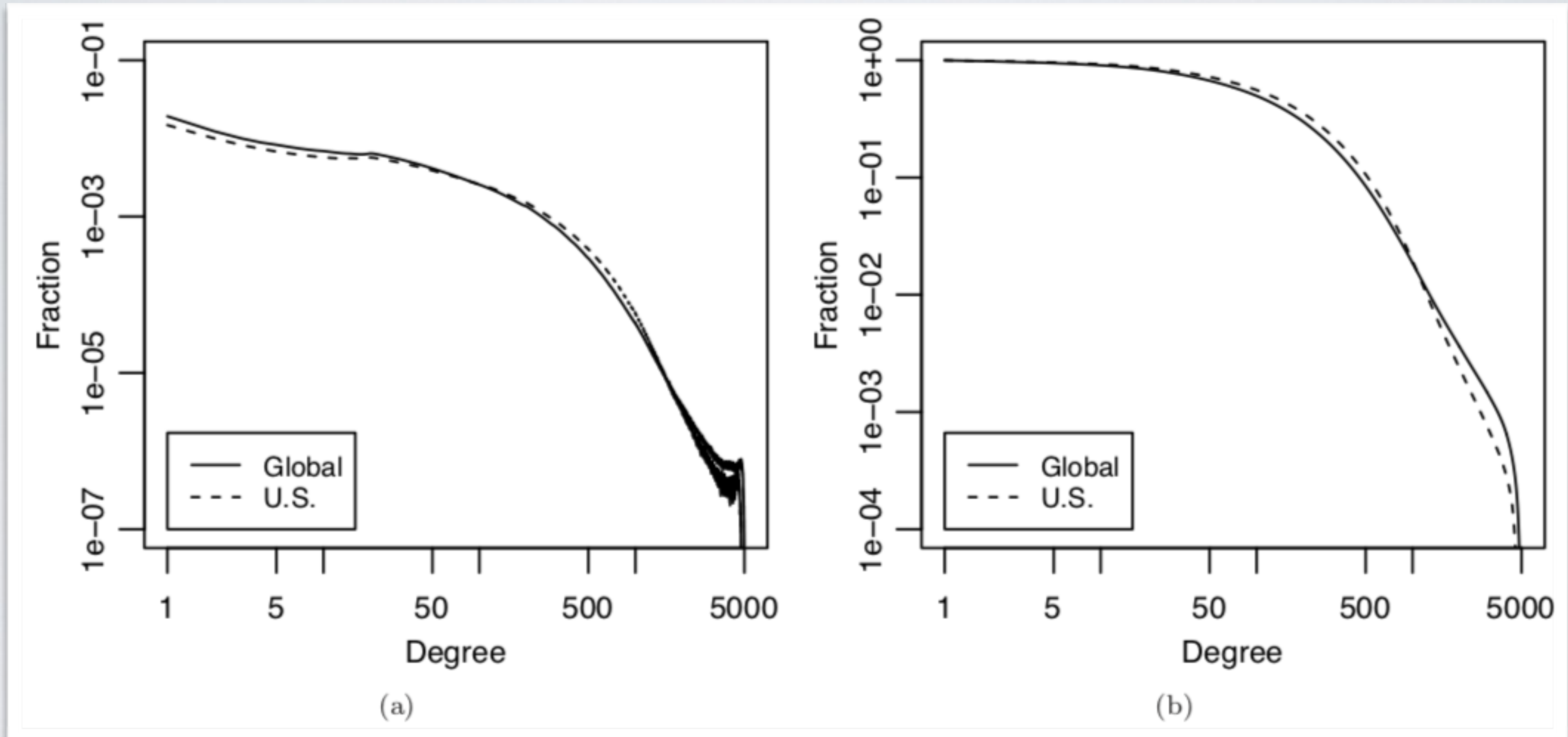
- 721M users (nodes) (active in the last 28 days)
- 68B edges
- Average degree: 190 (average # friends)
- Median degree: 99
- Connected component: 99.91%

EXEMPLE OF GRAPH ANALYSIS



Component size
Distribution

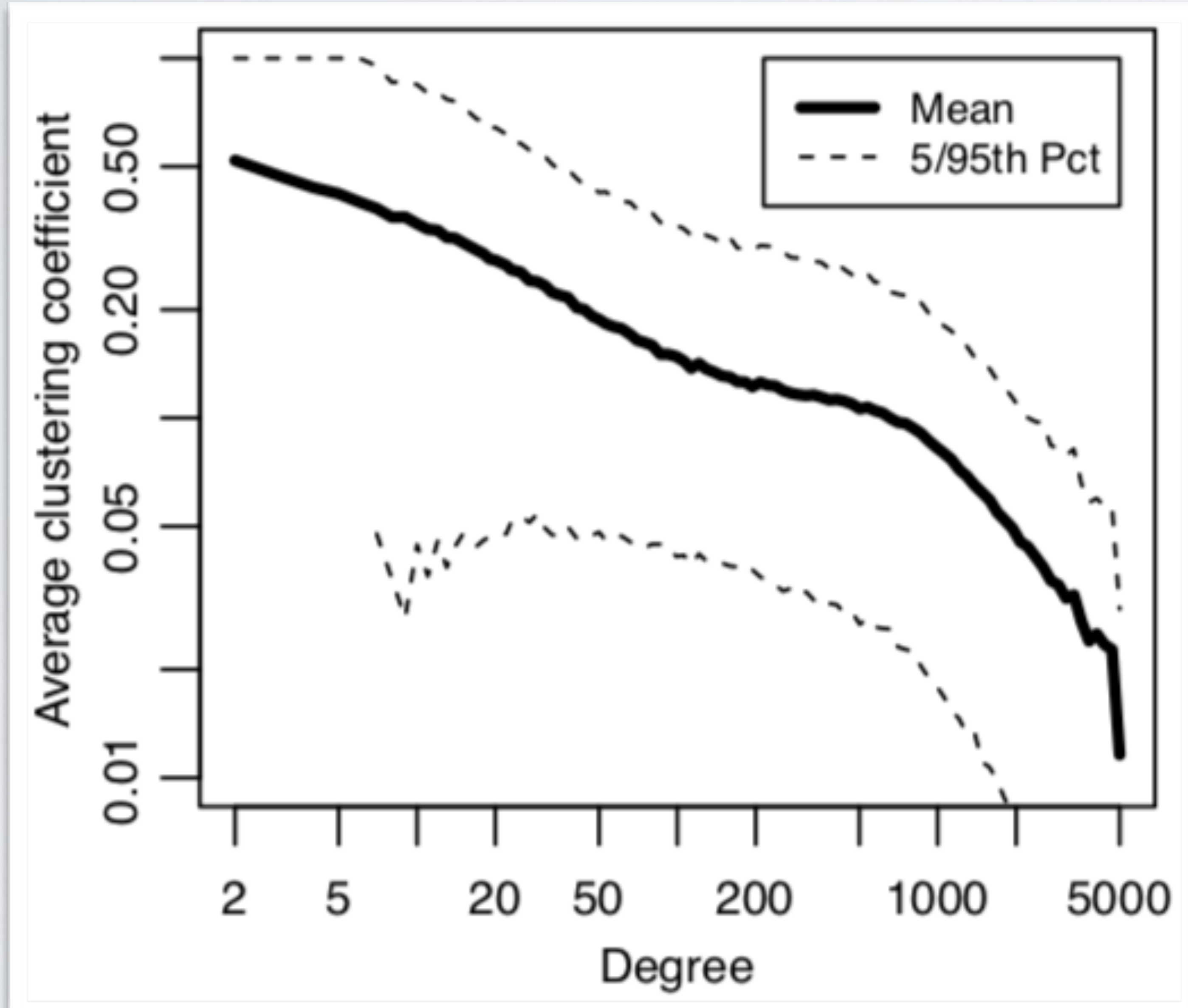
EXEMPLE OF GRAPH ANALYSIS



Cumulative

Degree distribution

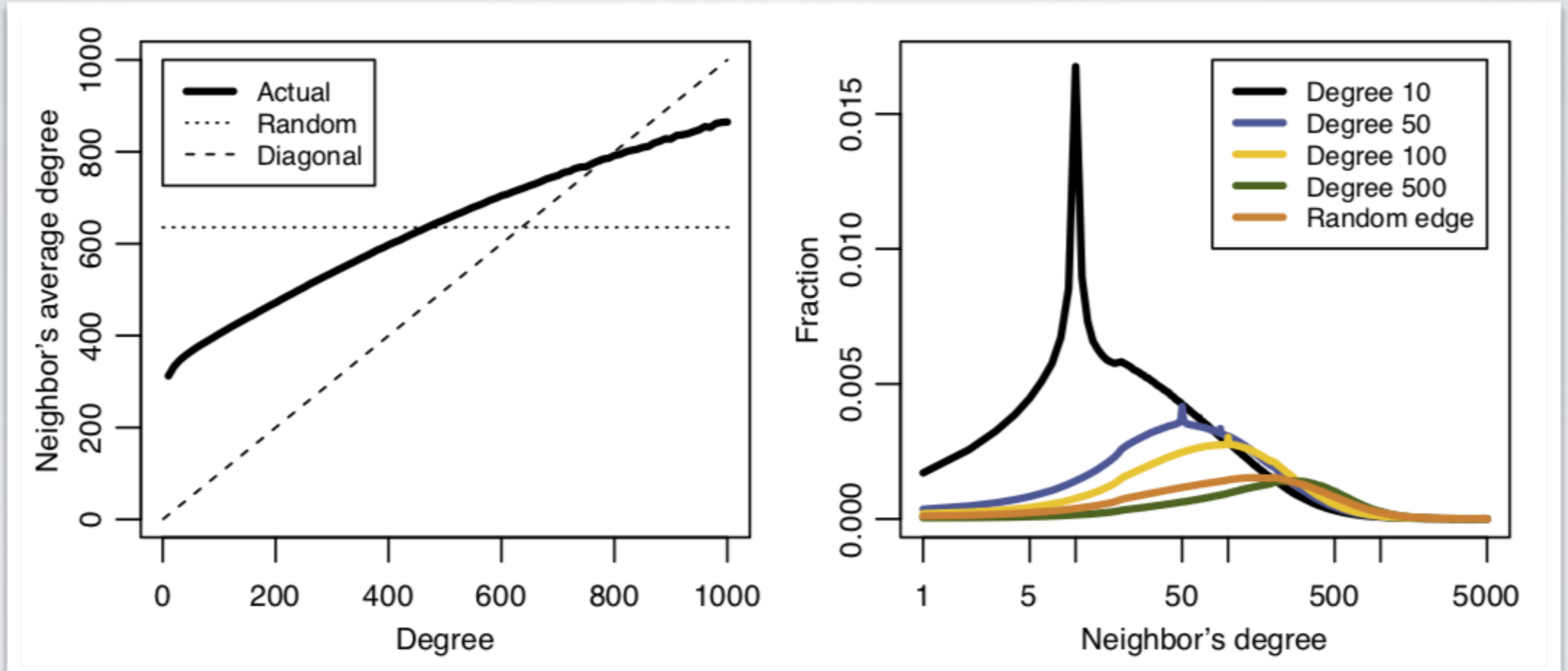
EXEMPLE OF GRAPH ANALYSIS



Clustering coefficient
By degree

Median user: 0.14:
14% of users with a common friend are friends

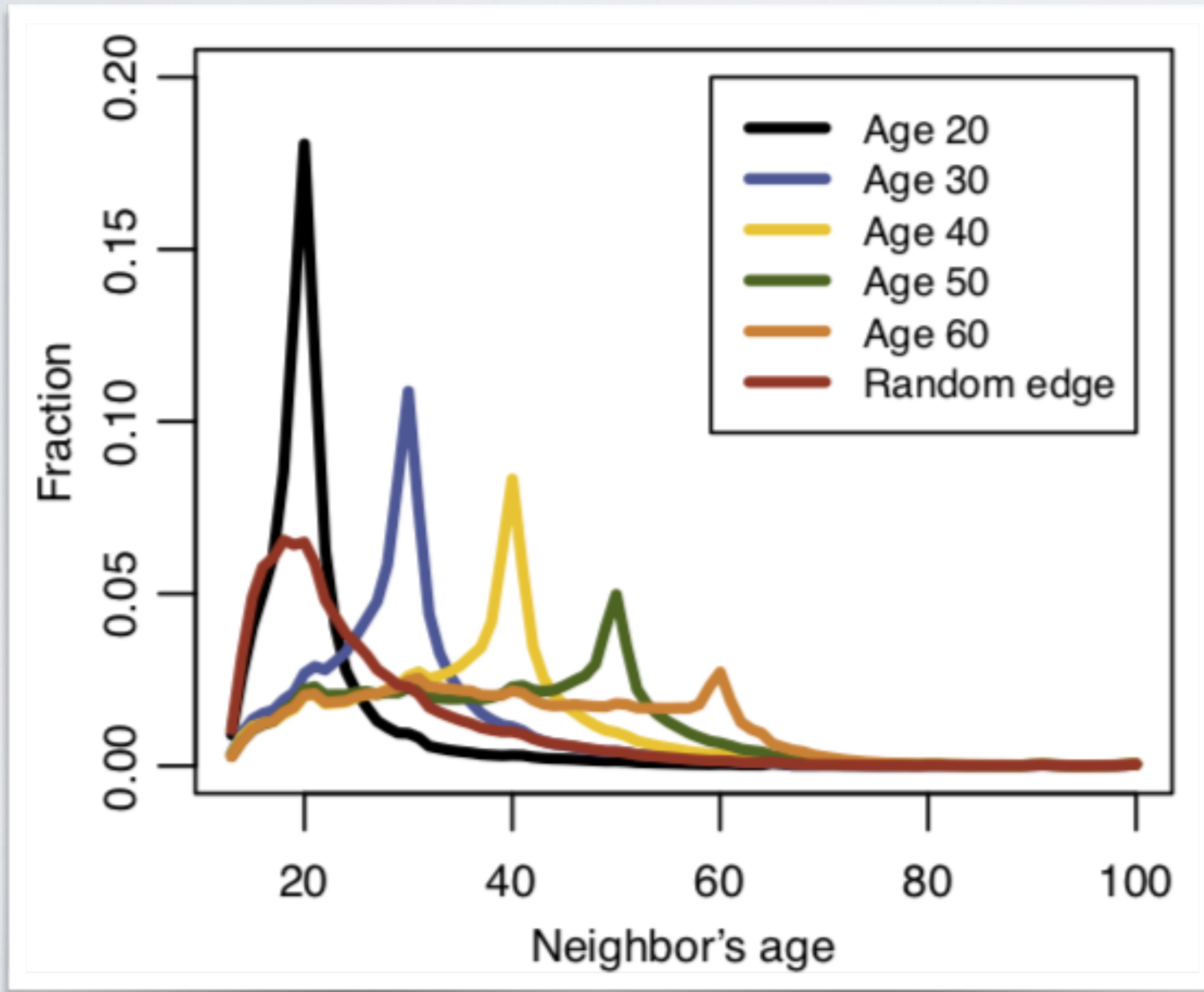
EXEMPLE OF GRAPH ANALYSIS



My friends have more Friends than me!

Many of my friends have the Same # of friends than me!

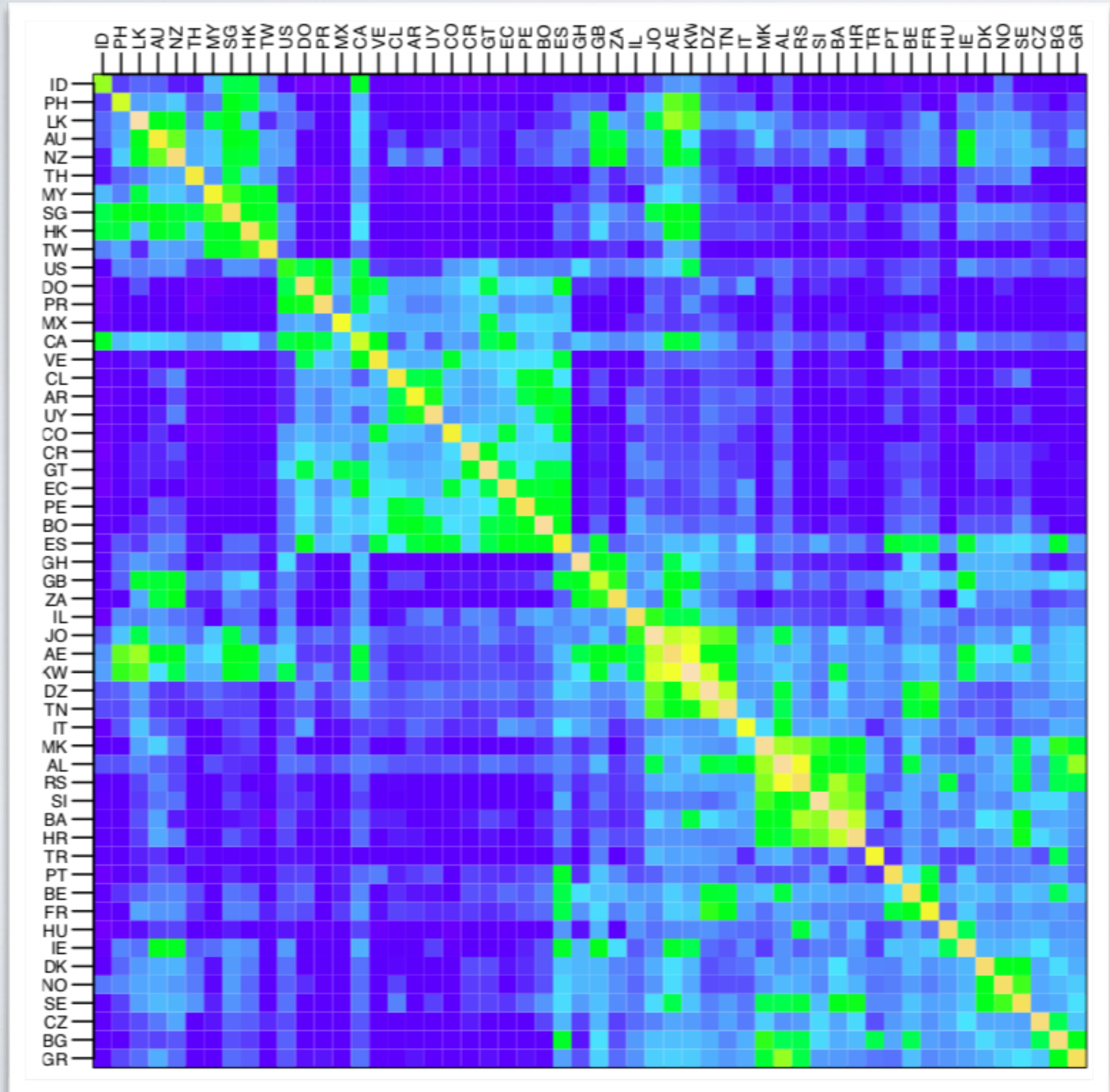
EXEMPLE OF GRAPH ANALYSIS



Age homophily

(More next class)

EXEMPLE OF GRAPH ANALYSIS



Country similarity

84.2% percent of edges are within countries

(More in the community detection class)