RANDOM GRAPHS MODELS

WHY USING RANDOM GRAPH MODELS

- Several good reasons:
  - Study some properties in a "controlled environment"
    - How does property X behaves when increasing property Y ?
  - Compare an observed network with a randomized version
    - Is observed property X "exceptional", or any similar network with same property Y and Z ?
  - Explain a given phenomenon
    - Such simple mechanism can reproduce property X and Y
  - Generate synthetic datasets
    - Testing an algorithm on 100 variations of the same network

CLASSES OF SYNTHETIC NETWORKS

Synthetic networks types

There are three main types of synthetic networks:

- **Deterministic models** are instances of famous graphs or, more commonly, repeated regular patters. e.g.,*Caveman graph, grids, lattices*.
- **Generative models** assign to each pair of nodes a probability of having an edge according to their properties (degree, label, etc.). e.g., *Erdős Rényi, Configuration model, etc.*
- **Mechanistic models** create networks by following a set of rules, a process defined by an algorithm. e.g., *Preferential attachment, Forest fire, etc.*

Fundamental network models

Central quantities in network analysis

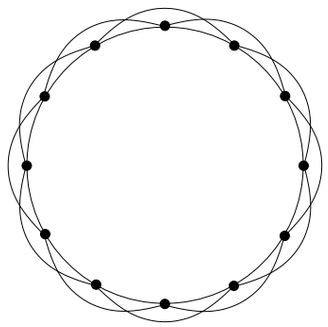
- Degree distribution: $P(k)$
- Clustering coefficient: C
- Average path length: $\langle d \rangle$

Network	Degree distribution	Path length	Clustering coefficient
Real world networks	broad	short	large

Regular lattices

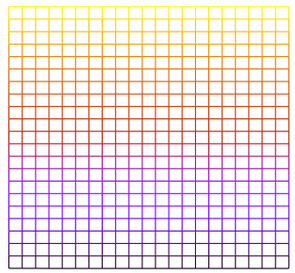
- Graphs where each node has the **same degree** k
- Translational symmetry in n directions

1D

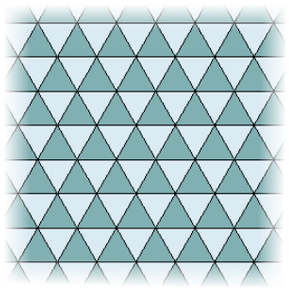


k=4

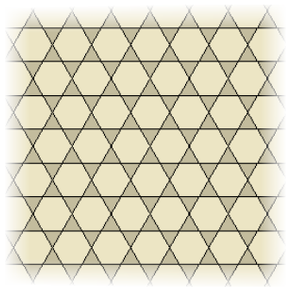
2D lattices



k=4

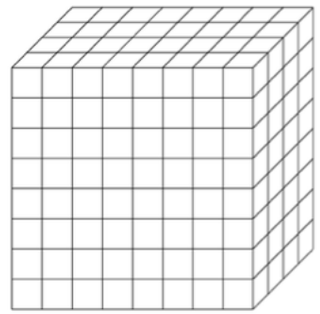


k=6

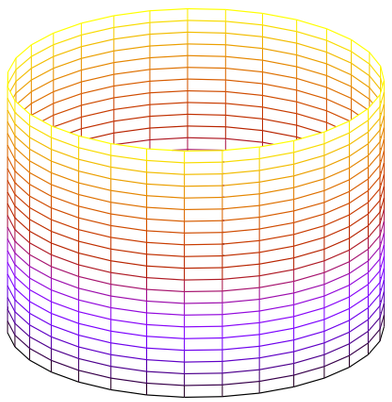


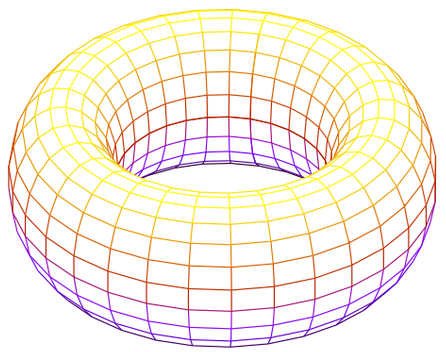
k=4

3D lattices



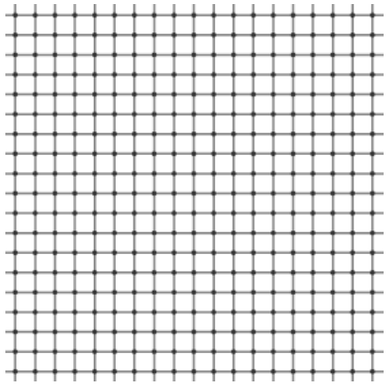
k=6



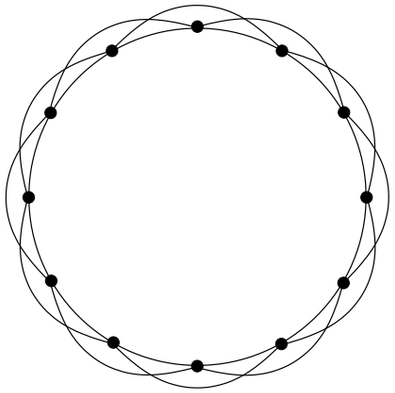


Regular lattices

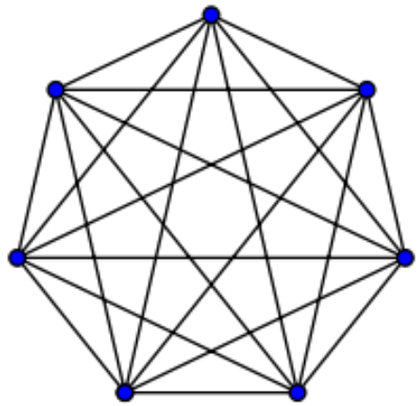
Clustering coefficient



C=0



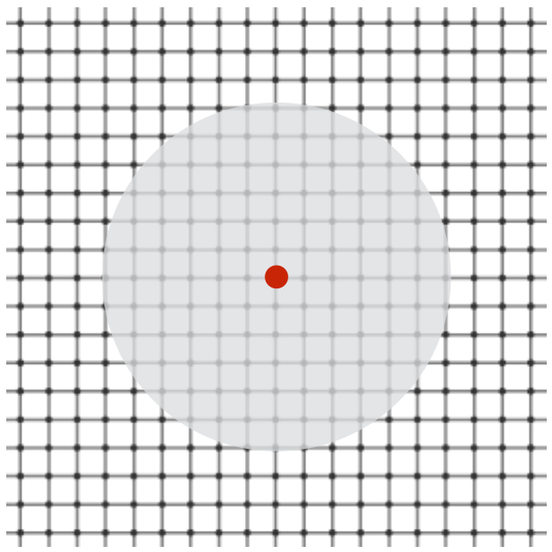
C=3/6



C=1

- Clustering coefficient depends on the structure (can be large or not)
- It is constant for each node

Path length



- Average path length grows quickly with n when k << n
- In a *large* graph with *realistic* average degrees, will be large

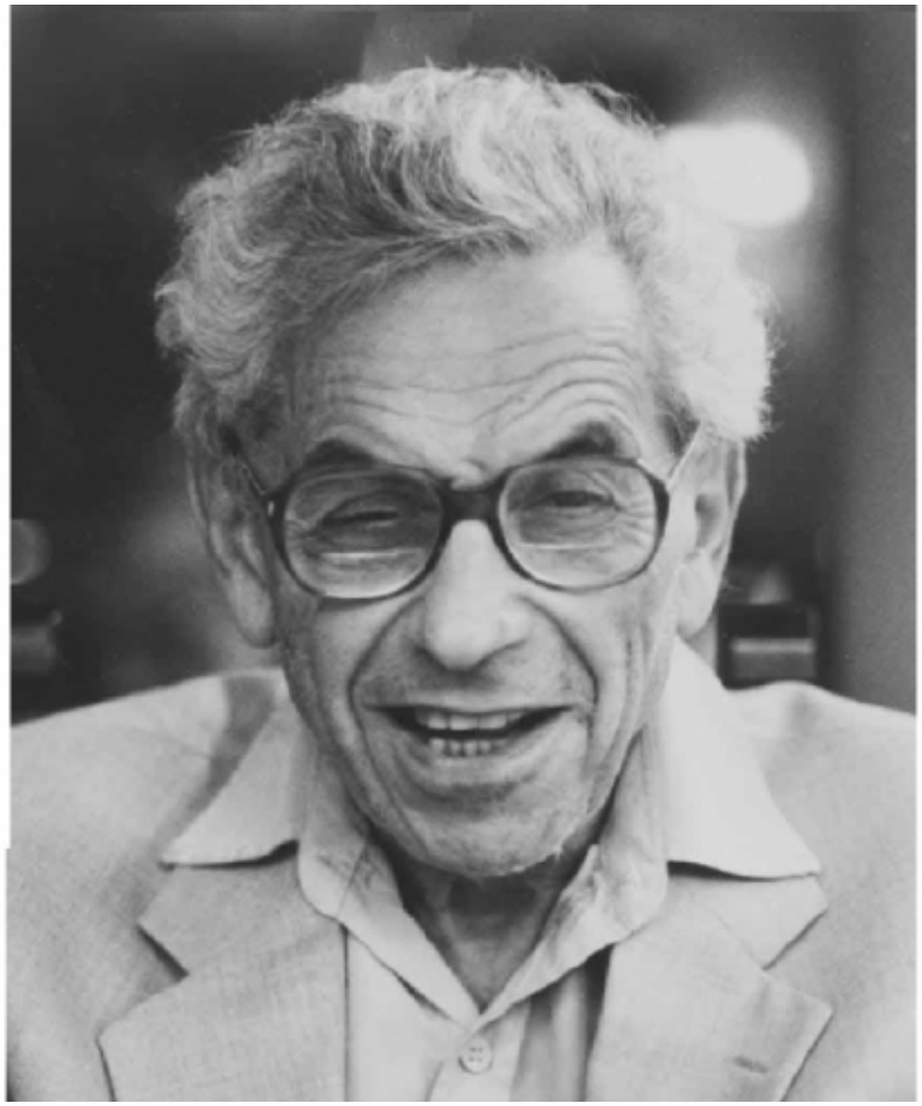
Regular lattices

Network	Degree distribution	Path length	Clustering coefficient
Real world networks	broad	short	large
Regular lattices	constant	long	can be large

PROBABILISTIC MODEL

The Erdős-Rényi Random Graph model (ER)

Random Graphs



Pál Erdős
(1913-1996)

Alfréd Rényi
(1921-1970)

"If we do not know anything else than the number *n* of nodes and the number *L* of links, the simplest thing to do is to put the links at random (no correlations)"

P. Erdős and A. Rényi. On random graphs, I. Publicationes Mathematicae (Debrecen), 6:290-297, 1959.
P. Erdős and A. Rényi. On the evolution of random graphs. Publ. Math. Inst. Hung. Acad. Sci., 5:17-61, 1960.

ER Random Graphs

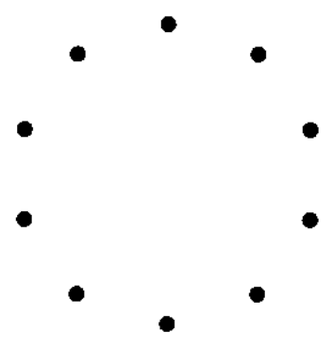
Erdős-Rényi model: simple way to generate random graphs

- The $G(n,L)$ definition
  1. Take n disconnected nodes
  2. Add L edges uniformly at random

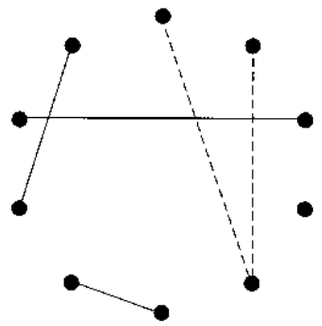
  Alternatively:
  - pick uniformly randomly a graph from the set of all graphs with n nodes and L links

- The $G(n,p)$ definition
  1. Take n disconnected nodes
  2. Add an edge between any of the nodes independently with probability p

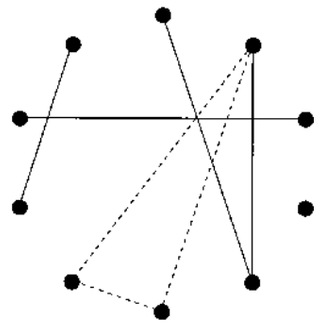
  Alternatively:
  - pick with probability $p^L(1-p)^{\binom{n}{2}-L}$ a network from the set of all networks with size n



p=0



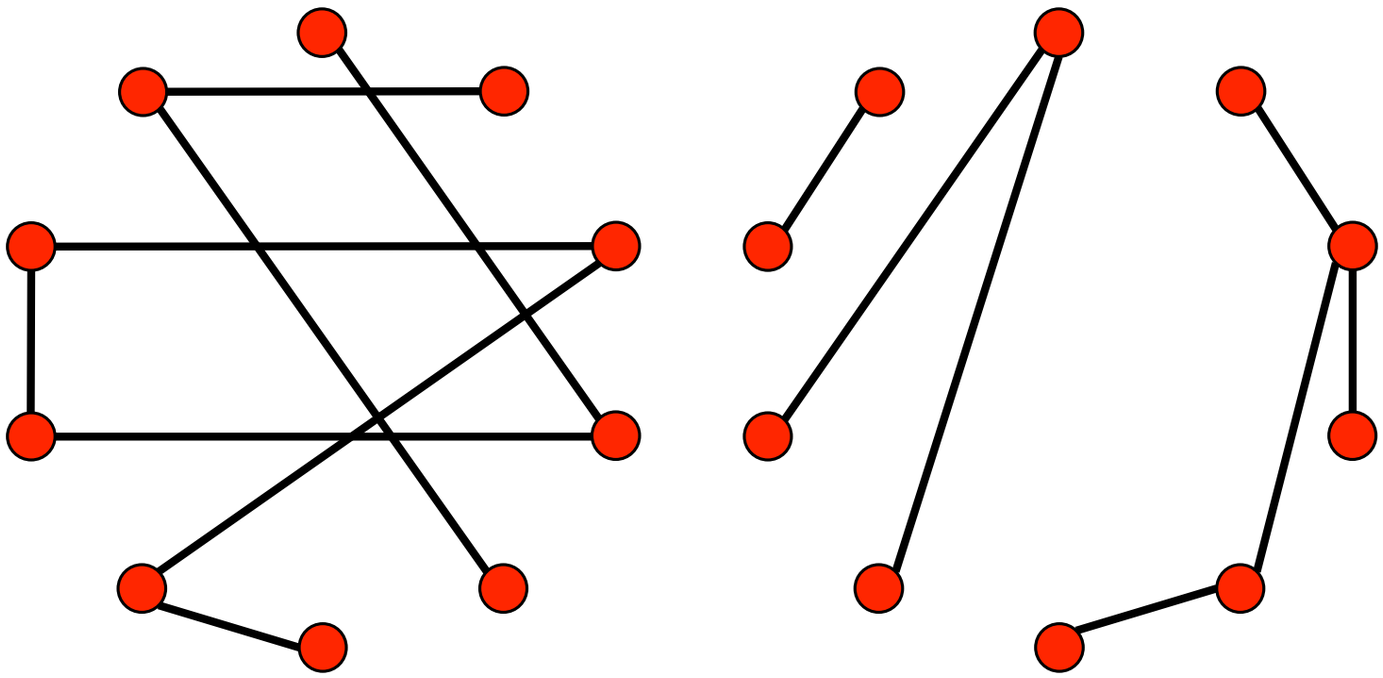
p=0.1



p=0.15

Random Graphs

In the G(n,p) variant, the number of edges may vary

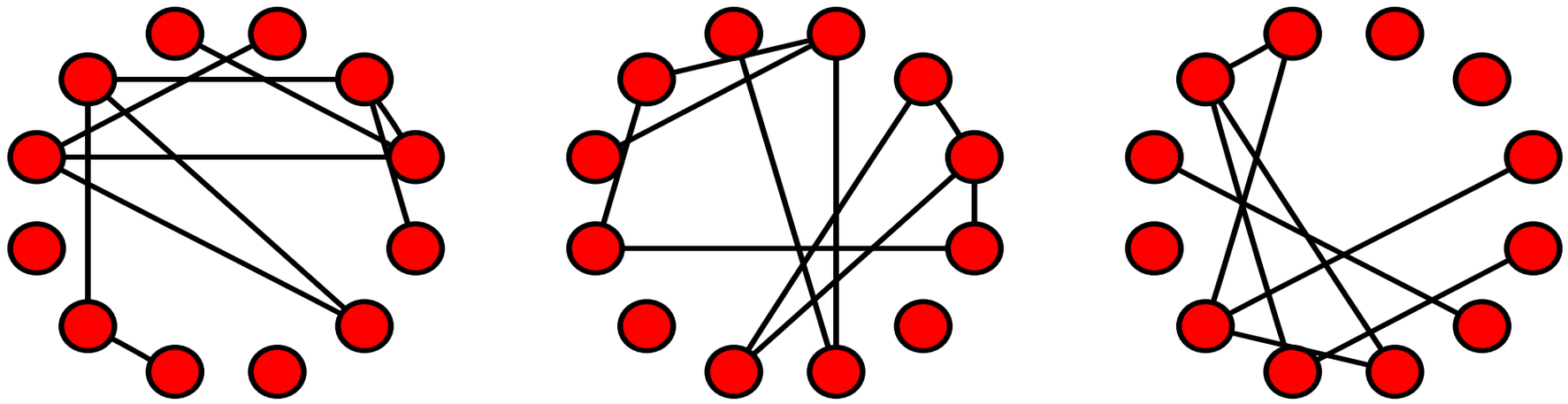


n=10

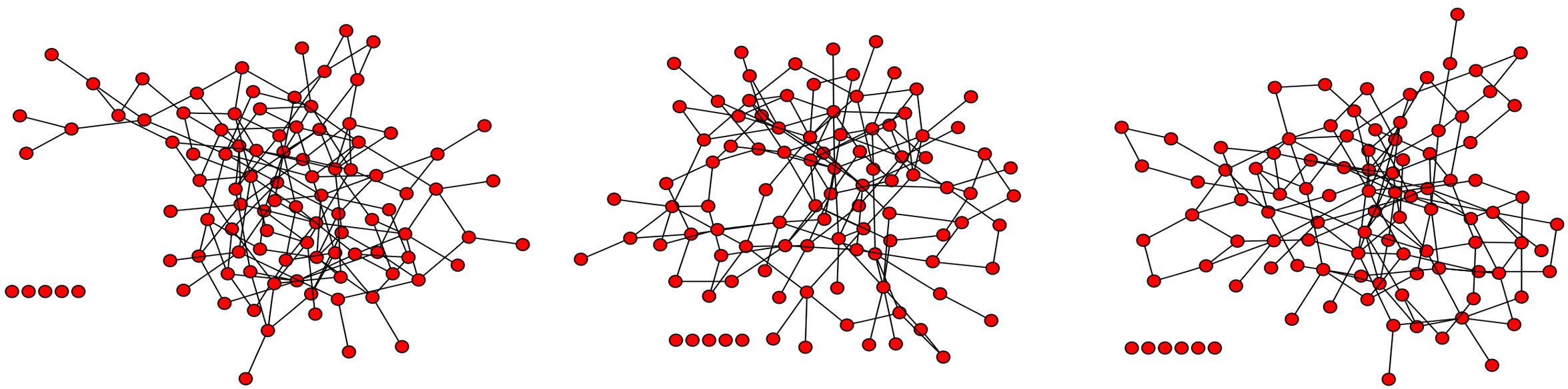
p=1/6

ER Random Graphs

p=1/6
N=12



p=0.03
N=100

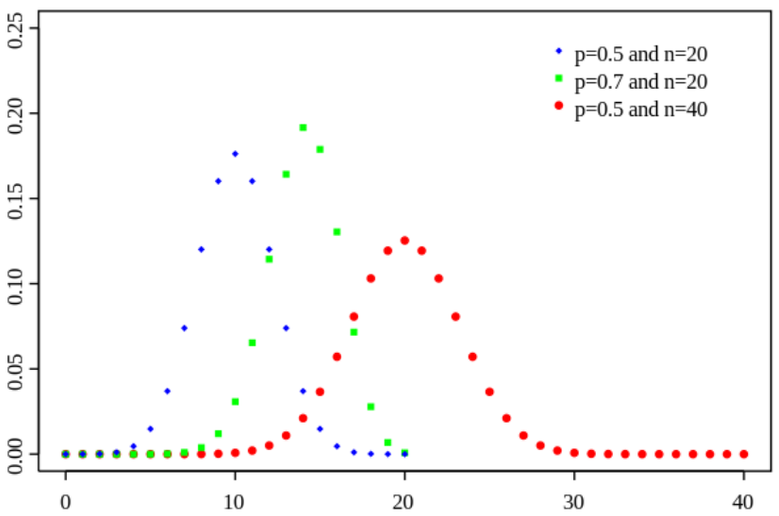


Random Graphs

P(L): probability to have exactly **L** links in a network of **n** nodes and probability **p**

Binomial distribution:

Discrete probability distribution of the number of successes(**x**) in a sequence of **N** independent experiments, with success probability **p**



$$P(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

Binomial coefficient:

Number of ways, disregarding order, that **k** objects can be chosen from among **n** objects

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Random Graphs

Binomial distribution

$$P(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

N: Number of experiments

Pairs of nodes

$$N = \binom{n}{2} = \frac{n(n-1)}{2}$$

P(L): probability to have exactly **L** links in a network of **n** nodes (with **p** the probability to have an edge)

$$P(L) = \binom{\binom{n}{2}}{L} p^L (1-p)^{\binom{n}{2}-L}$$

Random Graphs

Properties of Binomial distribution

Definition $$P(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

Mean $$< x > = pN$$

variance $$\sigma^2 = Np(1-p)$$

Random Graphs

Expected number of links <L>

$$< L > = pN = p\frac{n(n-1)}{2}$$

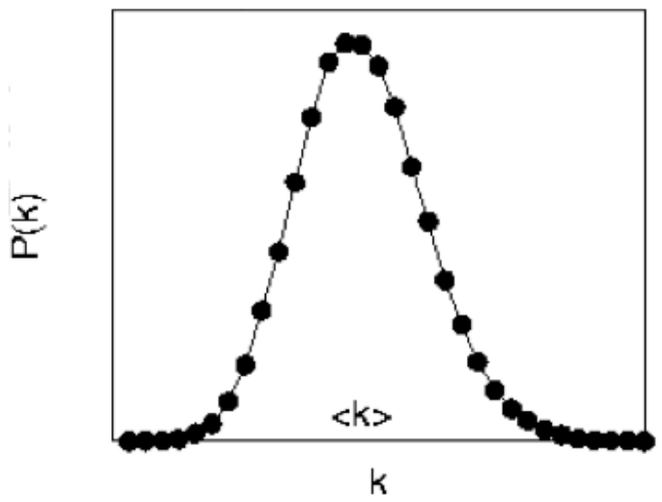
Expected average degree <k>

$$< k > = 2L/n = p(n-1)$$

Variance

$$\sigma^2 = Np(1-p) = \frac{n(n-1)}{2}p(1-p)$$

Degree distribution - Random Graphs



For each node,
independent probabilities to take each neighbor
=> Binomial distribution

P(k): probability to have exactly **k** links among **n** (total # of nodes), with **p** the (overall) probability to have an edge

$$P(k) = \binom{n-1}{k} p^k (1-p)^{(n-1)-k}$$

Characteristics:

$$< k > = p(n-1)$$

$$\sigma_k^2 = p(n-1)(1-p)$$

$$\frac{\sigma_k}{< k >} = \left[\frac{1-p}{p}\frac{1}{(N-1)}\right]^{1/2} \approx \frac{1}{(N-1)^{1/2}}$$

As the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of <k>.

Degree distribution - Random Graphs

For large **n** and small **k** (p,L), we can approximate the degree distribution using a poisson distribution of parameter (mean) $\lambda = < k >$

Poisson distribution

$$P(K) = \frac{\lambda^K e^{-\lambda}}{K!}$$

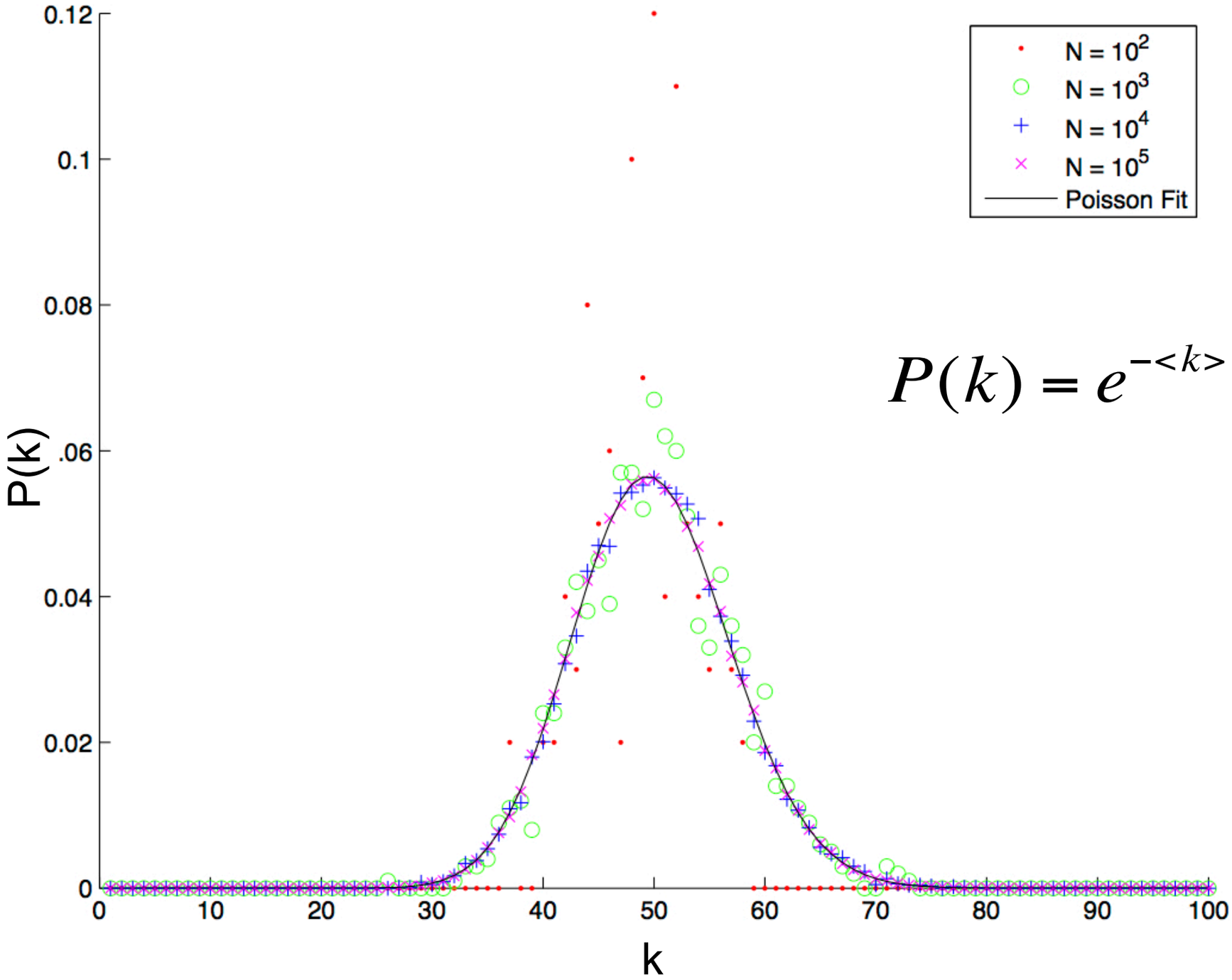
Distribution of degrees

$$P(k) = \frac{< k >^k \, e^{-<k>}}{k!}$$

standard deviation

$$\sigma = \sqrt{< k >}$$

Degree distribution - Random Graphs



$$P(k) = e^{-<k>} \frac{< k >^k}{k!}$$

Degree distribution - Random Graphs

Conclusion: degree distribution is **not**

-Heterogeneous

-Long tail

-Scale free

Clustering - Random Graphs

Local clustering of a node

Reminder, clustering coefficient

$$C_i \equiv \frac{2n_i}{k_i(k_i-1)}$$ where n_i is the number of links between the neighbours of node i

- Edges are independent and have the same probability p

$$n_i \cong p\,\frac{k_i(k_i-1)}{2}$$

possible links btw neighbors

$$p = \frac{<k>}{n\text{-}1}$$

$$C_i = \frac{2<k>}{n\text{-}1}\ \frac{k_i\,(k_i\text{-}1)}{2}\ \frac{1}{k_i\,(k_i\text{-}1)} = \frac{<k>}{n\text{-}1} = p$$

- For fixed average degree C is decreasing as N goes large

  ➡ Low clustering coefficient

  ➡ It is vanishing with the system size

Clustering - ER Random Networks

- **Small clustering coefficient**

$$C_i \equiv \frac{1}{N} < k >= p$$

Real-world networks

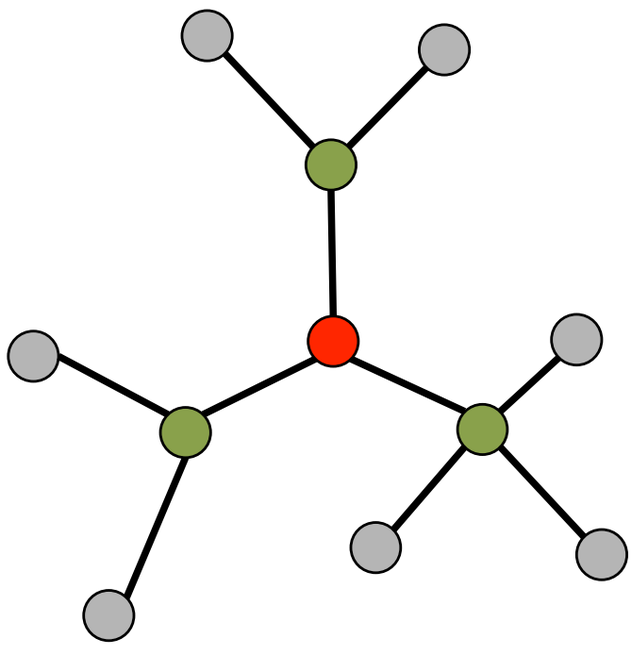
Network	Size	$\langle k \rangle$	ℓ	ℓ_{rand}	C	C_{rand}	Reference
WWW, site level, undir.	153 127	35.21	3.1	3.35	0.1078	0.00023	Adamic, 1999
Internet, domain level	3015–6209	3.52–4.11	3.7–3.76	6.36–6.18	0.18–0.3	0.001	Yook *et al.*, 2001a, Pastor-Satorras *et al.*, 2001
Movie actors	225 226	61	3.65	2.99	0.79	0.00027	Watts and Strogatz, 1998
LANL co-authorship	52 909	9.7	5.9	4.79	0.43	1.8×10^{-4}	Newman, 2001a, 2001b, 2001c
MEDLINE co-authorship	1 520 251	18.1	4.6	4.91	0.066	1.1×10^{-5}	Newman, 2001a, 2001b, 2001c
SPIRES co-authorship	56 627	173	4.0	2.12	0.726	0.003	Newman, 2001a, 2001b, 2001c
NCSTRL co-authorship	11 994	3.59	9.7	7.34	0.496	3×10^{-4}	Newman, 2001a, 2001b, 2001c
Math. co-authorship	70 975	3.9	9.5	8.2	0.59	5.4×10^{-5}	Barabási *et al.*, 2001
Neurosci. co-authorship	209 293	11.5	6	5.01	0.76	5.5×10^{-5}	Barabási *et al.*, 2001
E. coli, substrate graph	282	7.35	2.9	3.04	0.32	0.026	Wagner and Fell, 2000
E. coli, reaction graph	315	28.3	2.62	1.98	0.59	0.09	Wagner and Fell, 2000
Ythan estuary food web	134	8.7	2.43	2.26	0.22	0.06	Montoya and Solé, 2000
Silwood Park food web	154	4.75	3.40	3.23	0.15	0.03	Montoya and Solé, 2000
Words, co-occurrence	460.902	70.13	2.67	3.03	0.437	0.0001	Ferrer i Cancho and Solé, 2001
Words, synonyms	22 311	13.48	4.5	3.84	0.7	0.0006	Yook *et al.*, 2001b
Power grid	4941	2.67	18.7	12.4	0.08	0.005	Watts and Strogatz, 1998
C. Elegans	282	14	2.65	2.25	0.28	0.05	Watts and Strogatz, 1998

Albert, R. et.al. Rev. Mod. Phy. (2002)

Distance - Random Graphs

low clustering coefficient=>

Random graphs tend to have a tree-like topology with almost constant node degrees.



- nr. of first neighbors: $N(u)_1 = < k >$
- nr. of second neighbors: $N(u)_2 = < k >^2$
- nr. of neighbours at distance d: $N(u)_d = < k >^d$

Intuition: At which distance are all nodes reached?

$$n = < k >^d \Rightarrow \log_{<k>} n = d \Rightarrow d = \frac{\log n}{\log < k >}$$

Diameter, avg. distance in $\mathcal{O}(\log n)$

Distance - ER Random Networks

- **Logarithmically short distance among nodes**

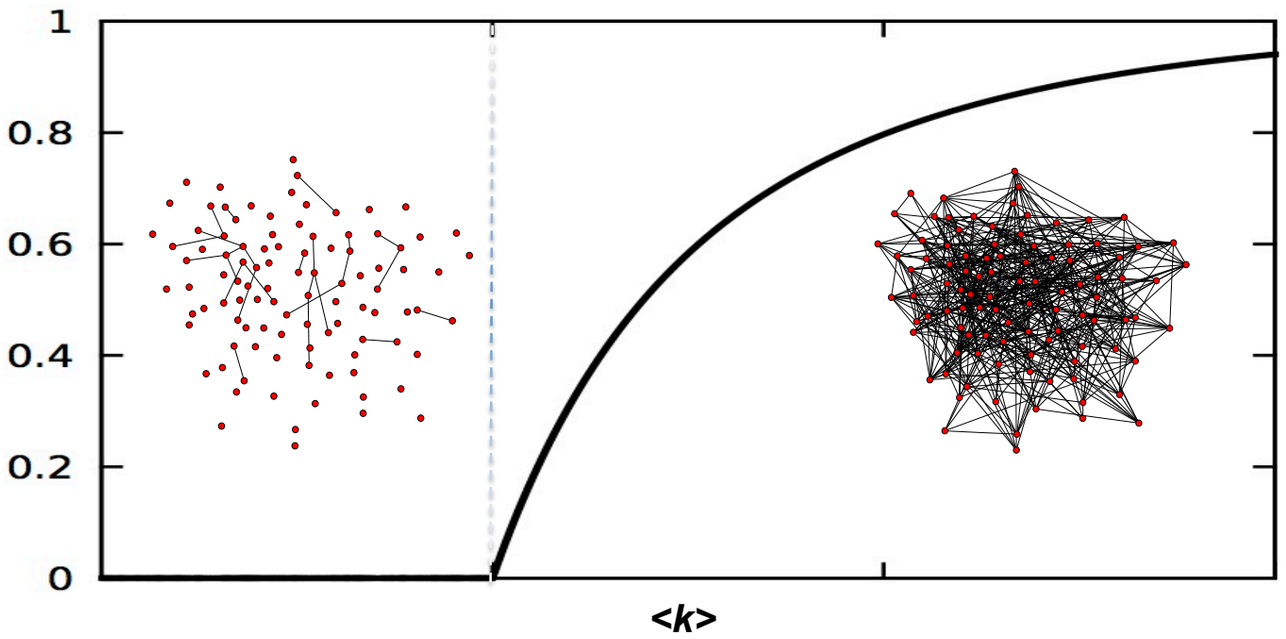
$$d = \frac{\log n}{log\langle k\rangle}$$

Real-world networks

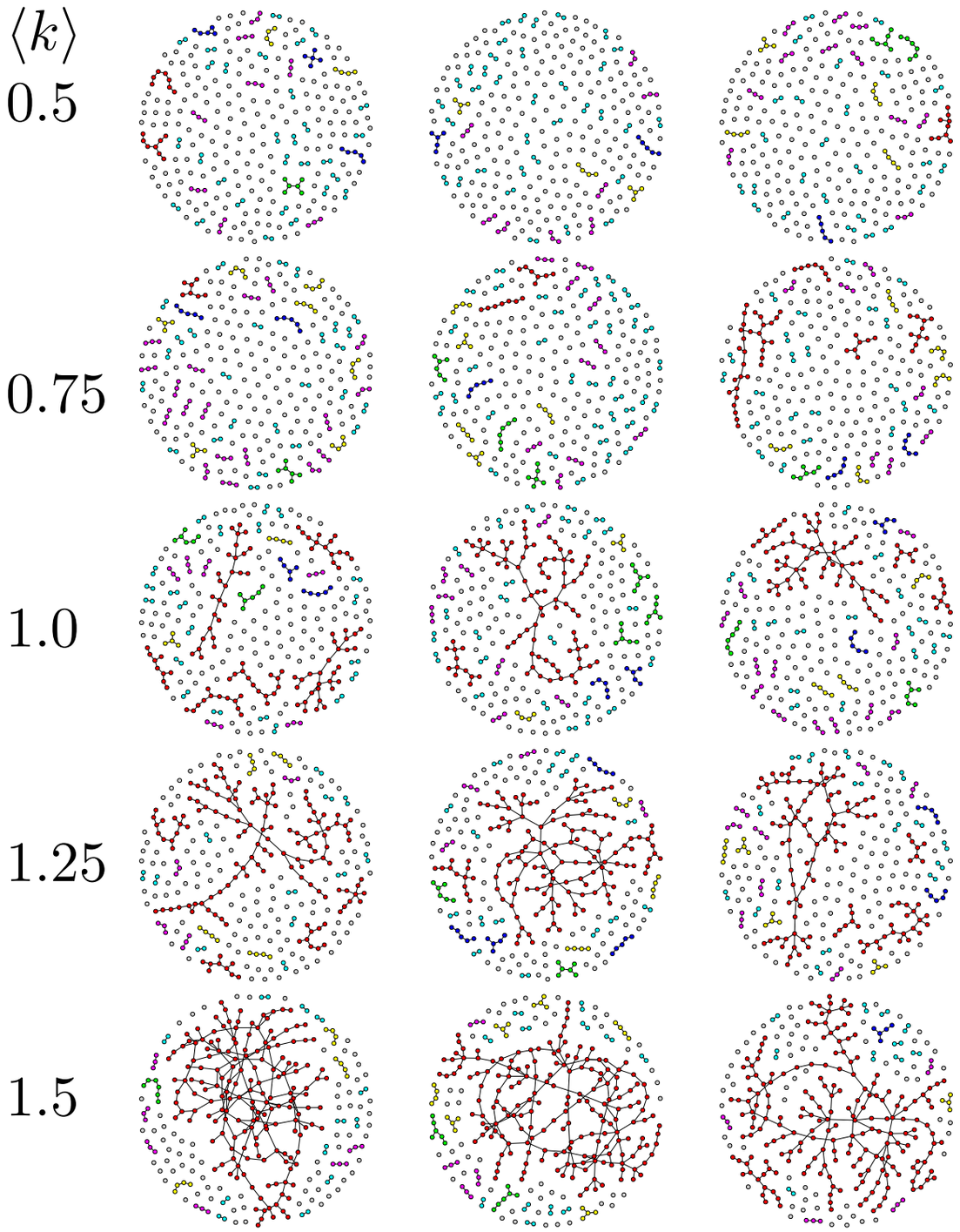
Network	Size	$\langle k\rangle$	ℓ	ℓ_{rand}	C	C_{rand}	Reference
WWW, site level, undir.	153 127	35.21	3.1	3.35	0.1078	0.00023	Adamic, 1999
Internet, domain level	3015–6209	3.52–4.11	3.7–3.76	6.36–6.18	0.18–0.3	0.001	Yook *et al.*, 2001a, Pastor-Satorras *et al.*, 2001
Movie actors	225 226	61	3.65	2.99	0.79	0.00027	Watts and Strogatz, 1998
LANL co-authorship	52 909	9.7	5.9	4.79	0.43	1.8×10^{-4}	Newman, 2001a, 2001b, 2001c
MEDLINE co-authorship	1 520 251	18.1	4.6	4.91	0.066	1.1×10^{-5}	Newman, 2001a, 2001b, 2001c
SPIRES co-authorship	56 627	173	4.0	2.12	0.726	0.003	Newman, 2001a, 2001b, 2001c
NCSTRL co-authorship	11 994	3.59	9.7	7.34	0.496	3×10^{-4}	Newman, 2001a, 2001b, 2001c
Math. co-authorship	70 975	3.9	9.5	8.2	0.59	5.4×10^{-5}	Barabási *et al.*, 2001
Neurosci. co-authorship	209 293	11.5	6	5.01	0.76	5.5×10^{-5}	Barabási *et al.*, 2001
E. coli, substrate graph	282	7.35	2.9	3.04	0.32	0.026	Wagner and Fell, 2000
E. coli, reaction graph	315	28.3	2.62	1.98	0.59	0.09	Wagner and Fell, 2000
Ythan estuary food web	134	8.7	2.43	2.26	0.22	0.06	Montoya and Solé, 2000
Silwood Park food web	154	4.75	3.40	3.23	0.15	0.03	Montoya and Solé, 2000
Words, co-occurrence	460.902	70.13	2.67	3.03	0.437	0.0001	Ferrer i Cancho and Solé, 2001
Words, synonyms	22 311	13.48	4.5	3.84	0.7	0.0006	Yook *et al.*, 2001b
Power grid	4941	2.67	18.7	12.4	0.08	0.005	Watts and Strogatz, 1998
C. Elegans	282	14	2.65	2.25	0.28	0.05	Watts and Strogatz, 1998

Albert, R. et.al. Rev. Mod. Phy. (2002)

Connected components of Random Graphs



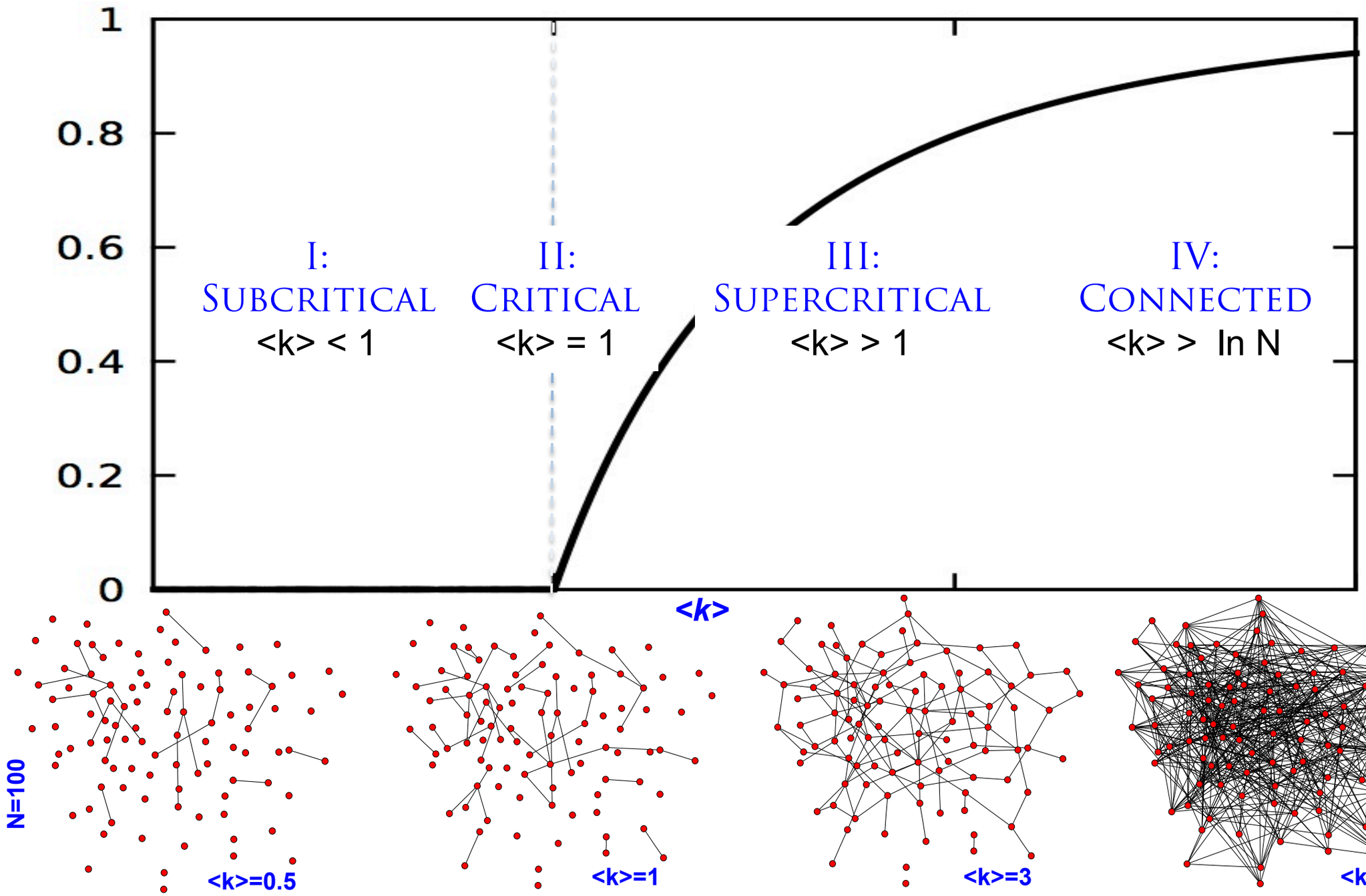
- Network structure goes through a transition
- Question: How and when does this transition happen



Connected components of Random Graphs

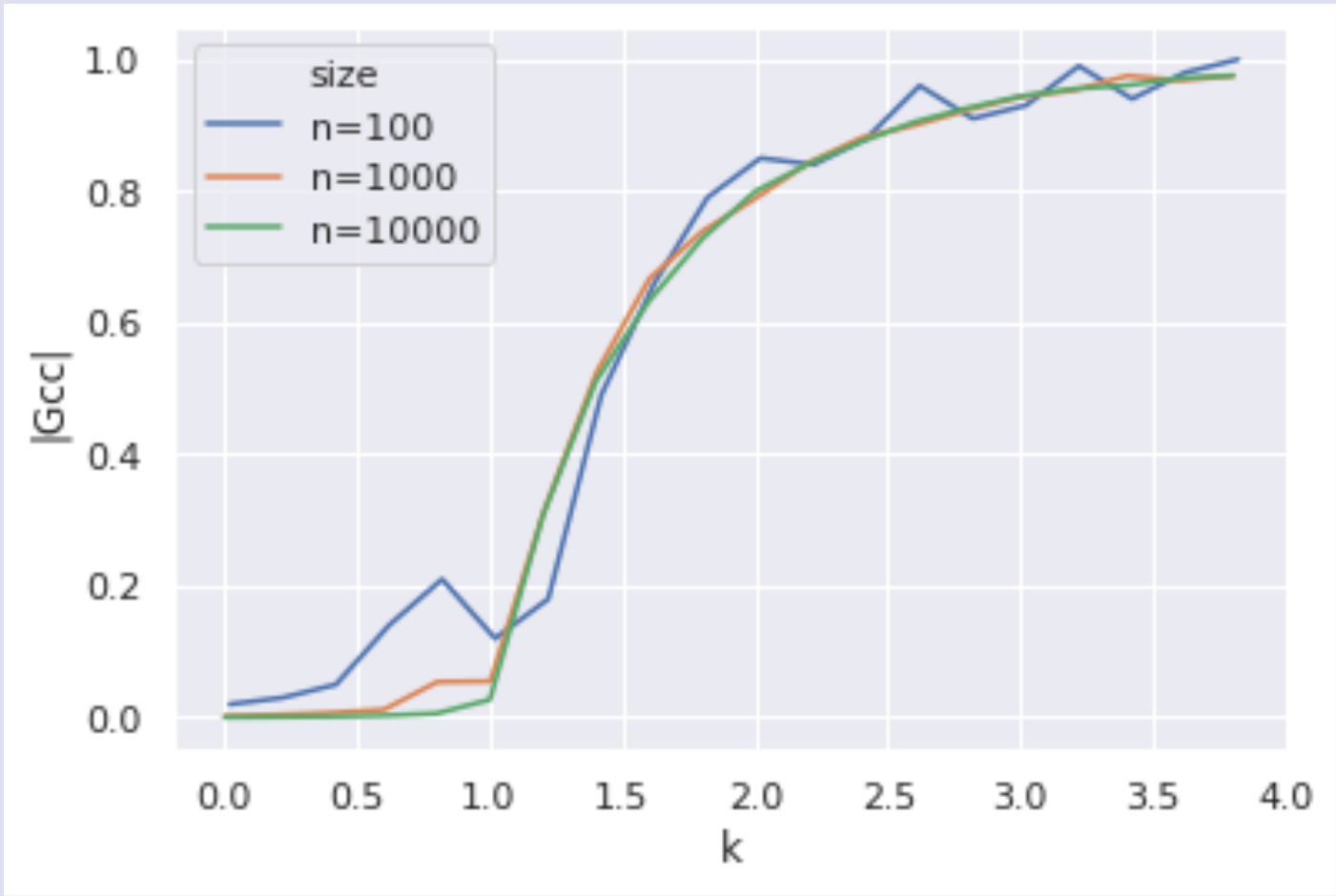
https://www.complexity-explorables.org/explorables/the-blob/

ER Random Network - catch up



Structural (percolation) phase transition at <k>=1 (or equivalently when p=1/N)

ER Random Network - catch up



An intuitive way to understand this phenomenon is to use the same observation of the graph being tree-like as previously. Since the number of nodes N that can be reached after d hops can be estimated to grow as $\langle k \rangle^d$, a value of $\langle k \rangle < 1$ leads to an impossibility to reach all nodes even for a large d, while $\langle k \rangle > 1$ leads to arbitrarily large N for long enough d. Proper demonstration and more details can be found in the original paper[a].

---

[a] Erdős and Rényi 1960.

ER Random Network - catch up

Basic characteristics

- Degree distribution

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k} \xrightarrow[N \to \infty]{} p_k = \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

Binomial distribution — Poisson distribution

Degree distribution without tail

- Clustering

$$C_i = \frac{<k>}{n\text{-}1} = p$$

Vanishing clustering coefficient for large size

- Path length

$$\mathcal{O}(\log n)$$

Distance with logarithmic relation to nodes

ER Random Network - catch up

Network	Degree distribution	Path length	Clustering coefficient
Real world networks	broad	short	large
Regular lattices	constant	long	large
ER random networks	Poissonian	short	small

It is not capturing the properties of any real system

BUT

it serves as a reference system for any other network model

Configuration model

More details at [http://tuvalu.santafe.edu/~aaronc/courses/5352/fall2013/csci5352_2013_L11.pdf]

Random graphs with specified degrees

Problem

- The ER Random Graph model has a Poisson degree distribution
- Most real-world networks have heavy-tailed degree distributions
- **We need to generate networks which have pre-determined degrees or degree distribution, but they are maximally random otherwise**
- The observed properties (clustering coefficient, etc.) might be due *only* to the difference in degree distribution

Random graphs with specified degrees

Configuration model

How much of some observed pattern is driven by the degrees alone?

Based on an observed network

- Defined as $G(n, \vec{k})$ where $\vec{k} = \{k_i\}$ is a degree sequence on n nodes, with k_i being the degree of node i

Ad hoc degree distribution

- **The degree sequence $\vec{k} = \{k_i\}$ can be sampled from a probability distribution**
  - Delta/Dirac function => Random regular graph
  - Poisson => Similar to ER for proper parameters
  - Scale-free => Power-law random graph
- Only global condition to satisfy is: $\sum_i k_i$ **mod** $2 = 0$

  (even degree sum) i.e. each edge has to have ending nodes

Random graphs with specified degrees

Configuration model

How much of some observed pattern is driven by the degrees alone?

Exact or approximate degree distribution

- The model can preserve the **expected** degree sequence, or the **exact** degree sequence
  - Chung-lu (appoximate)
  - Molloy-reed (Exact)

Random graphs with specified degrees

Chung-Lu model for configuration networks = Approximate degree distribution

- Probabilistic model which produce a network with degrees approximating (on average) the original degree
- It is a *"coin-flipping"* process as ER model but **the probability that two nodes i and j are connected depends on the degree k_i and k_j of the ending nodes**
- From the point of node i with degree k_i, the probability that one of its edges will connect to j with k_j:

$$k_j/2m$$

- This can happen via k_i links, thus the probability that they are connected:

$$p_{ij} = \frac{k_i k_j}{2m}$$

assuming that: $[\max(k_i)]^2 < 2m$

(/!\ inconsistent probability, it is rather expected number of edges)

- Chung-Lu model takes each pairs of nodes and connects them with this probability

$$\forall_{i>j} \qquad A_{ij} = A_{ji} = \begin{cases} 1 & \text{with probability } p_{ij} \\ 0 & \text{otherwise} \end{cases}$$

Random graphs with specified degrees

Chung-Lu model for configuration networks = Approximate degree distribution

$$\forall_{i>j} \qquad A_{ij} = A_{ji} = \begin{cases} 1 & \text{with probability } p_{ij} \\ 0 & \text{otherwise} \end{cases} \qquad \text{where} \qquad p_{ij} = \frac{k_i k_j}{2m}$$

- Each pairs of nodes are considered once, thus it produces a **simple graph** (without self-loops and multi edges)

- Degree of a node equals only in “expectation” to the originally assigned degree

- Inconsistency for large degrees in small networks $\quad [\max(k_i)]^2 < 2m$

Complexity:

- $O(n^2)$: We need $n(n-1)$ flips to test all node pairs

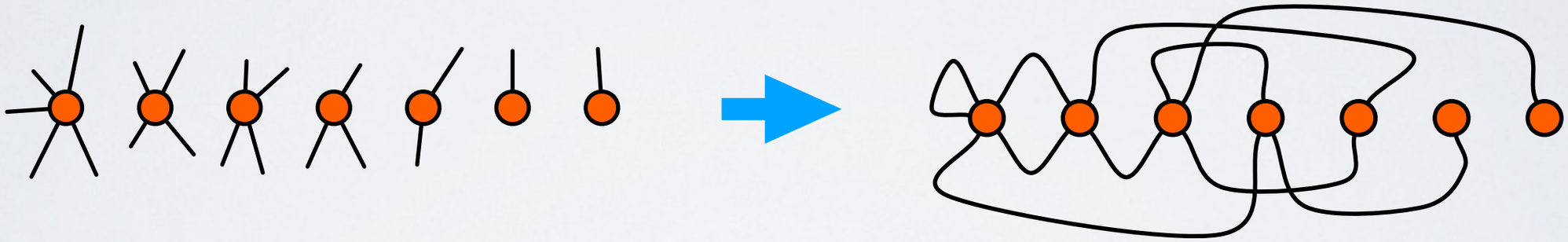
EXPENSIVE!

Random graphs with specified degrees

Molloy-Reed model for configuration networks = exact degree preservation

Original idea:

1. Given a degree sequence $\vec{k} = \{k_1, k_2, \ldots, k_n\}$
2. Assign each node $i \in V$ with k_i number of stubs
3. Select random pairs of unmatched stubs and connect them
4. Repeat 3 while there are unmatched stubs



- This process will produce a configuration model with exact degree sequence
- Possible to select multiple times stubs of the same pair of nodes ➜ **Multilinks**
- Possible to select the stubs of the same node to connect ➜ **Self-links**

The obtained graph is not simple…but the density of multi and self-links ➜ 0 as N ➜ ∞

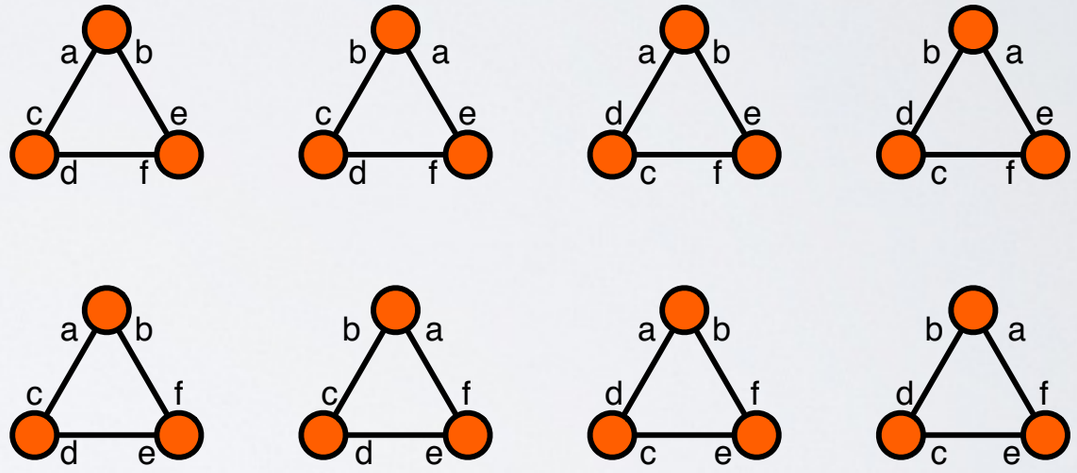
Random graphs with specified degrees

Molloy-Reed model for configuration networks = exact degree preservation

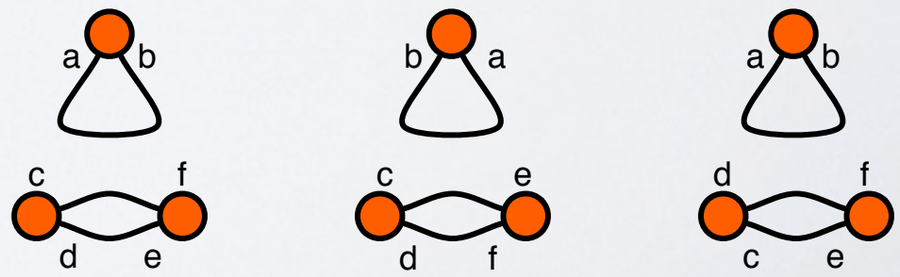
Non-unique problem

- Matching of stubs appears with equal probability
- **BUT networks with the same $\{k_i\}$ do not appear with equal probability**
- More than one matching can correspond to the same network (topologically)

Different matchings yield same graphs



Some graphs produced by less combinations =>less likely to appear

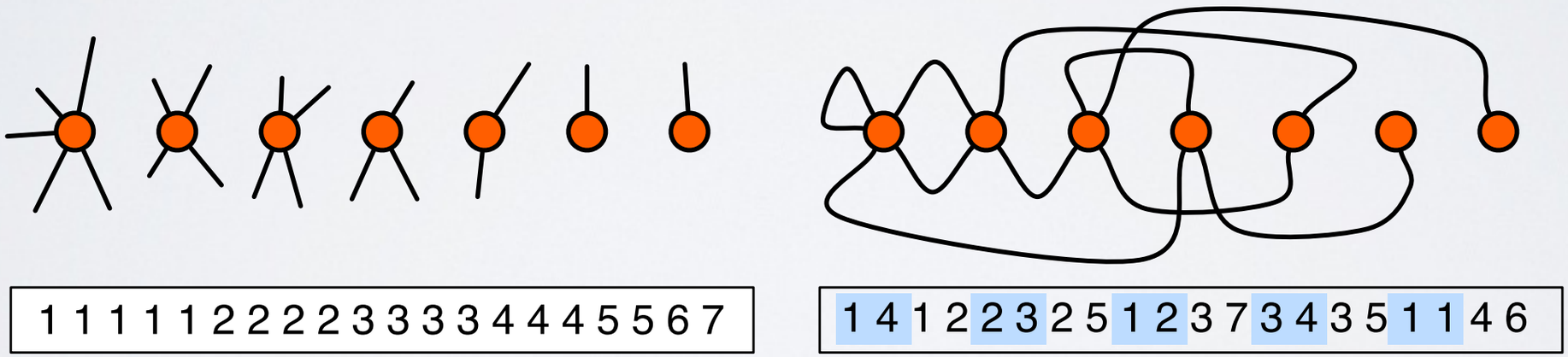


Random graphs with specified degrees

Molloy-Reed model for configuration networks = exact degree preservation

An effective algorithm:

1. Take an array $\vec{v}$ with length $2m$ and fill it with exactly k_i indices of each node $i \in V$
2. Make a random permutation of the array $\vec{v}$
3. Read the content of the array in an order and in pairs
4. Pairs of consecutive node indices will assign links in the configuration network



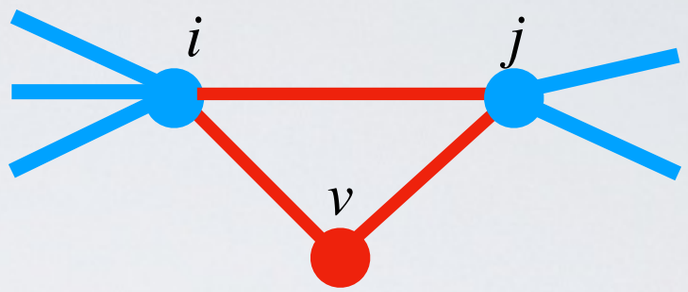
Complexity:

- *O(m)*: Random permutation of an array
- *O(m log m)*: assigning uniformly random variables to indices and quick-sort them

CHEAP!

Configuration model - mathematical properties

Expected clustering coefficient



It is the average probability that two neighbours of a vertex are neighbours

- Start at some vertex v (with degree $k \geq 2$)
- Choose a random pair of its neighbours i and j
- The probability that i and j are themselves connected is $k_i k_j / 2m$

Clustering coefficient

independent of network size

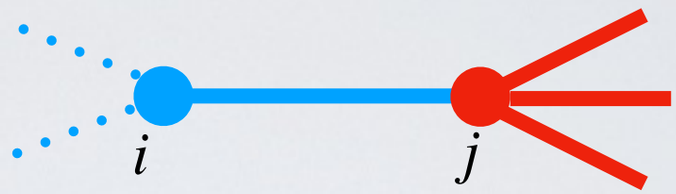
$$C = \ldots = \frac{1}{n} \frac{[\langle k^2 \rangle - \langle k \rangle]^2}{\langle k \rangle^3}$$

- It is a vanishing quantity $O(1/n)$ as long as the second moment is finite (not power law)

For details, see: http://tuvalu.santafe.edu/~aaronc/courses/5352/fall2013/csci5352_2013_L12.pdf

Configuration model - mathematical properties

Neighbors's degrees



What is the degree distribution of neighbors of a randomly chosen vertex?

- Let p_k be the fraction of vertices in the network with degree k
- There are np_k vertices of degree k in the network.
- The end point of every edge in the network has the same probability $\frac{k}{2m}$ of connecting to a vertex of degree *k*
- **Degree distribution of a randomly picked neighbor (of any node)**

$$p_{neighb,k} = \frac{k}{2m} np_k = \frac{kp_k}{\langle k \rangle}$$

Configuration model - mathematical properties

- Degree distribution of a randomly picked neighbor (of any node)

$$p_{neighb,k} = \frac{k}{2m} n p_k = \frac{k p_k}{\langle k \rangle}$$

- Average degree of a randomly picked neighbor

$$\langle k_{neighb} \rangle = \sum_k k p_{neighb,k} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

- Larger than $\langle k \rangle$ as soon as degrees are heterogeneous ➡ **Friendship paradox**

1 node with degree 10, 10 nodes with degree 1:

$$\langle k \rangle = \frac{10 + 1 * 10}{11} = 1.81..$$

$$\langle k^2 \rangle = \frac{10^2 + 1^2 * 10}{11} = 10$$

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{10}{1.82} = 5.5$$

ER Random Network - catch up

Network	Degree distribution	Path length	Clustering coefficient
Real world networks	broad	short	large
Regular lattices	constant	long	large
ER random networks	Poissonian	short	small
Configuration Model	Custom, can be broad	short	small