SPATIAL NETWORKS

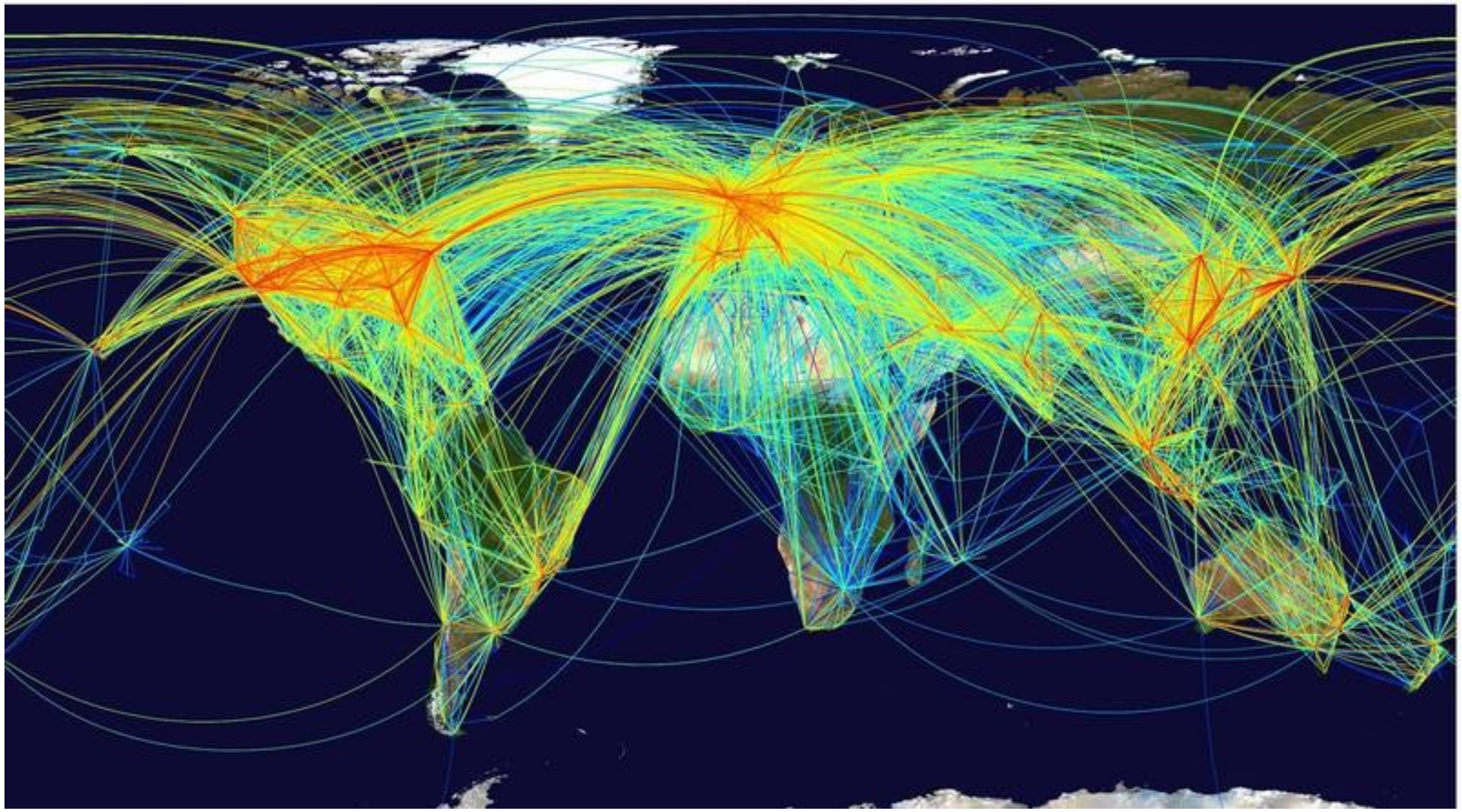
Spatial networks

A network is said spatial if the distance between nodes affect the probability of observing edges between them

- Can be seen as a special case of Assortativity, generalizing the notion of distance through several dimensions.

Distance

- Physical distance
- Economical distance
- Social distance
- Difference in professional categories
- …



Geisel et.al. (2011)

Literature

Physics Reports 499 (2011) 1–101

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Spatial networks

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A R T I C L E   I N F O

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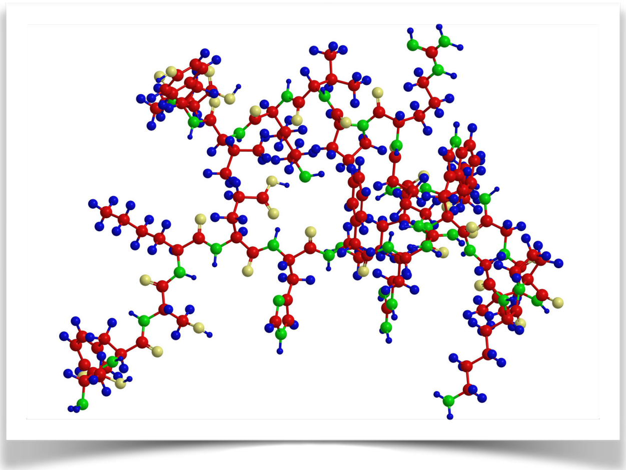
A B S T R A C T

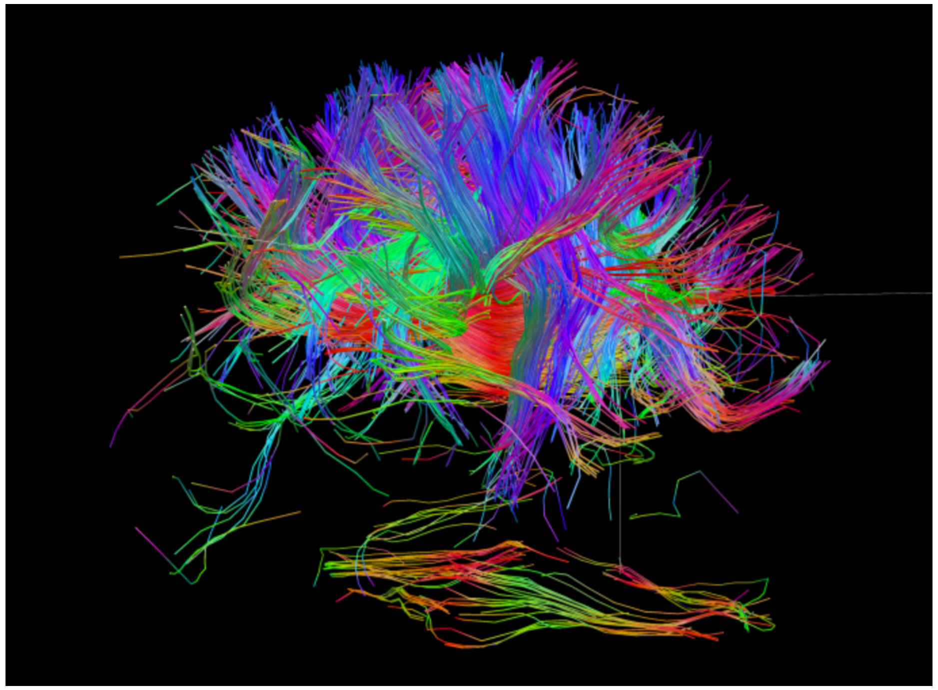
Complex systems are very often organized under the form of networks where nodes and edges are embedded in space. Transportation and mobility networks, Internet, mobile phone networks, power grids, social and contact networks, and neural networks, are all examples where space is relevant and where topology alone does not contain all the information. Characterizing and understanding the structure and the evolution of spatial networks is thus crucial for many different fields, ranging from urbanism to epidemiology.

Spatial networks

Types of spatial networks

- Transportation networks
  - Airline networks
  - Bus, subway, railway, and commuters
  - Cargo ship networks
- Infrastructure networks
  - Road and street networks
  - Power grids and water distribution networks
  - The internet
- Neural networks
- Protein networks
- Mobility networks
- Social networks
- …

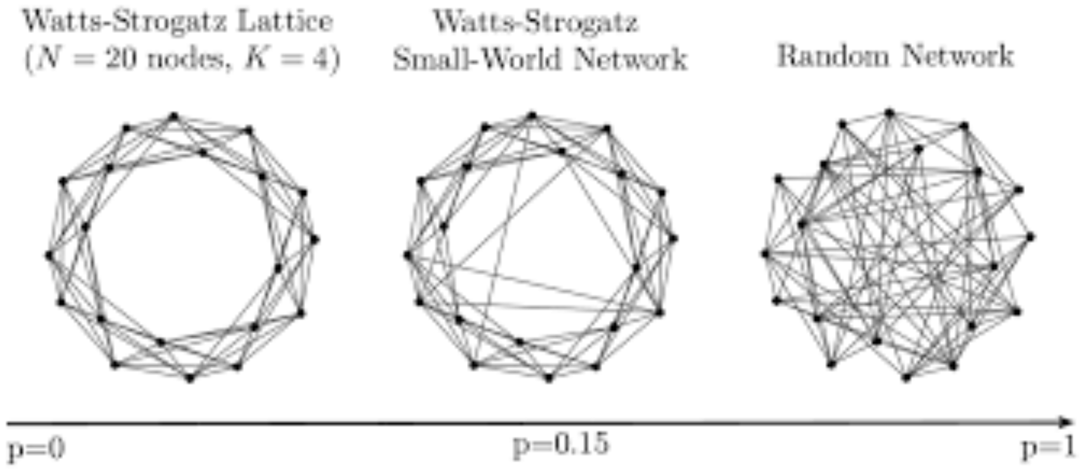


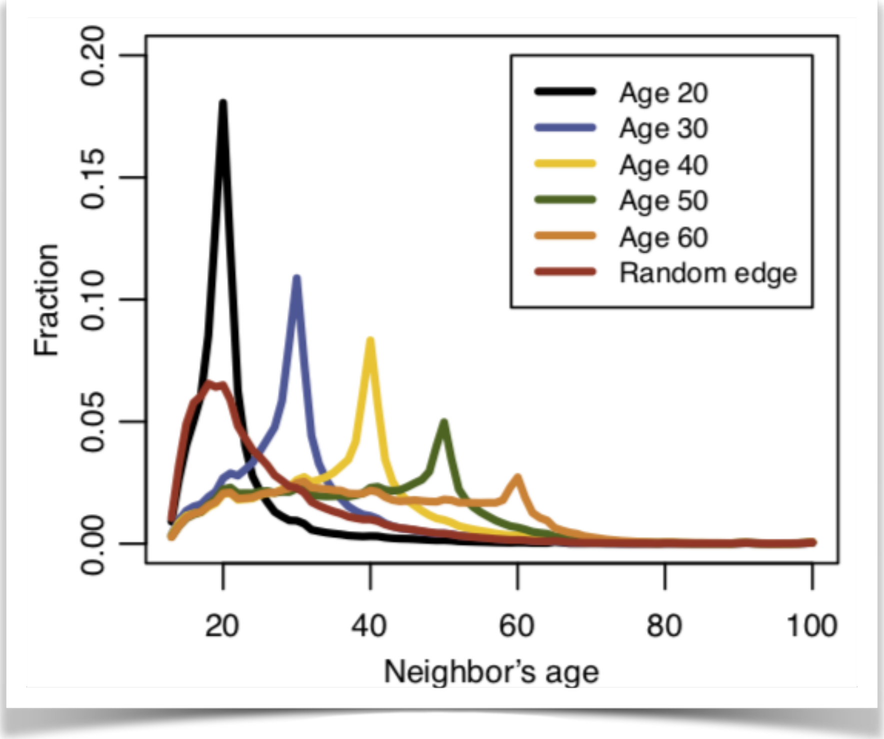


Spatial networks

Examples of 1D spaces

- The watts-Strogatz random graph is defined on a (circular) 1D space: each node is (initially) connected to its k closest nodes in this space.
- In social networks, users tend to be more connected with other users with similar age. We can consider age as a position on 1D space. The same is true about political opinions, if we consider a Left-Right spectrum.

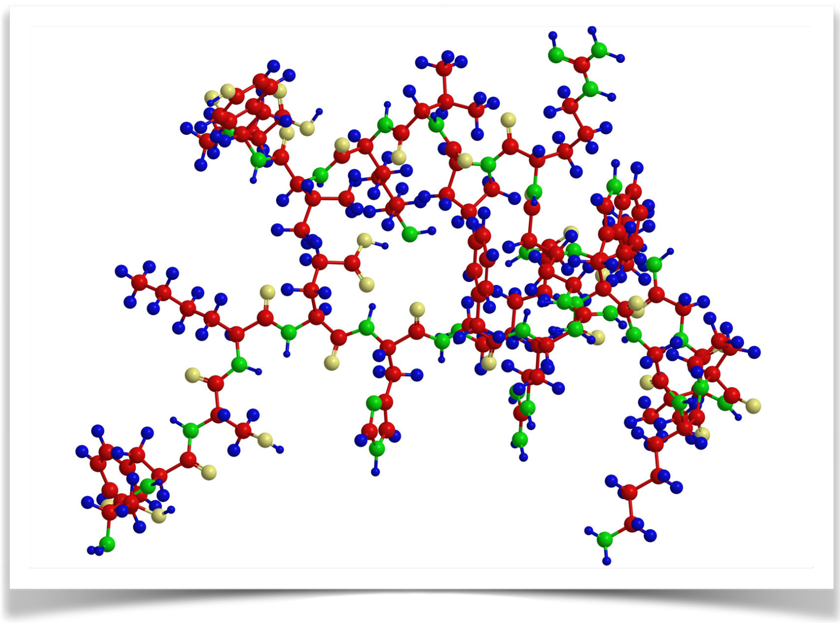


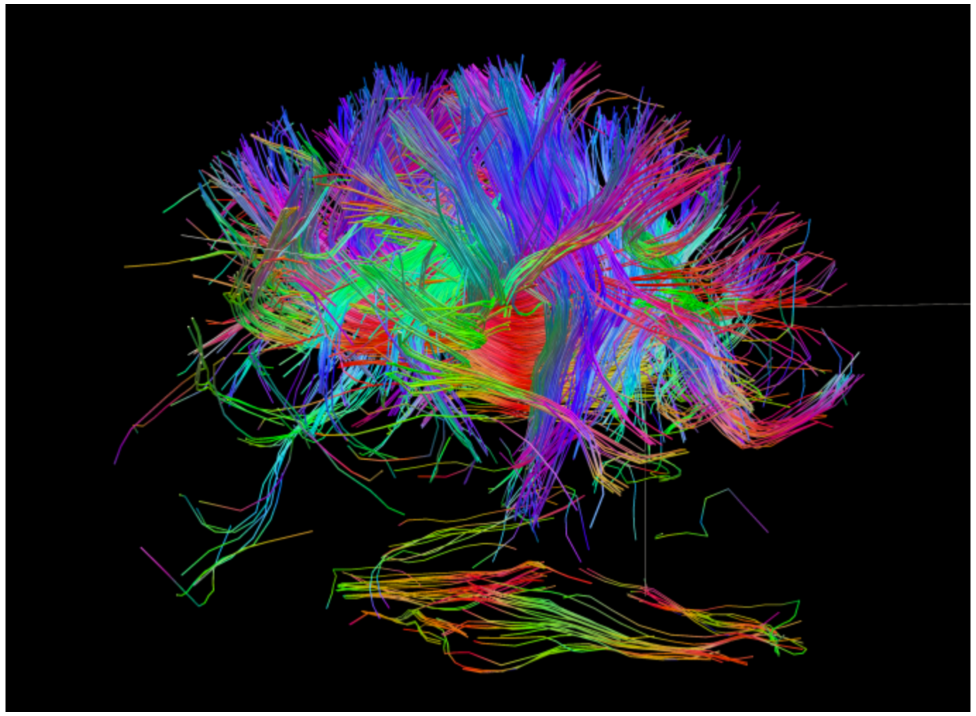


Spatial networks

Examples of 3+D spaces

- If we consider altitude, geographical networks are 3D spaces
- If we consider multiple nodes properties as dimensions, nodes can be located on high dimensional spaces, e.g., age, political opinion, revenue, geographical location, etc. Be careful however, that analyzing a spatial networks needs to define the *distance* between nodes, which can be tricky to define if dimensions are of different natures.
- Methods such as *graph embedding* assign locations in arbitrary large dimensions to nodes that summarize some of the network properties (see later class).





Spatial networks

Distances

The distance between each pair of nodes can be computed in different ways, depending on the nature of dimensions nodes are embedded in. The most common ones are:

- **Euclidean distance**, or $L^2 distance$ is the usual, straight line distance
- **Great-Circle distance** is used to measure the distance between points located on a sphere, typically the Earth for geographical data.
- **Dot product** and **Cosine Distance** are often used in high dimensions, in particular when it makes sense to multiply the location vectors.
- **Manhattan distance**, or $L^1 distance$, is sometimes used as a variant of Euclidean distance for high dimensional data (it is simply defined as the sum of differences in each of the dimensions.)
- Observed distance can sometimes be used, a typical example being **average time distance**: in datasets of trips or traffic, the time distance between dots might be only loosely proportional to geographical distance.

Spatial networks

Notation

Δ_{uv}	**Metric distance** between u and v (Euclidean, Manhattan, etc.)
ℓ_{uv}	**Route distance** between u and v, i.e., sum of Metric distances between nodes on the shortest path between u and v
s_u^{Δ}	**Distance strength**, cumulative distance from a node to its neighbors. $s_u^{\Delta} = \sum_{v \in N(u)} \Delta_{uv}$. The relation between k_u and s_u^{Δ} can be studied, for instance to see if larger nodes tend to connect at longer distance.

Spatial networks

Route factor - Accessibility

$Q(u,v)$ **Route Factor**, also called the detour index, measures how *efficiently* the network allows to go from a node to another, it is defined as the ratio between the metric distance and the route distance:

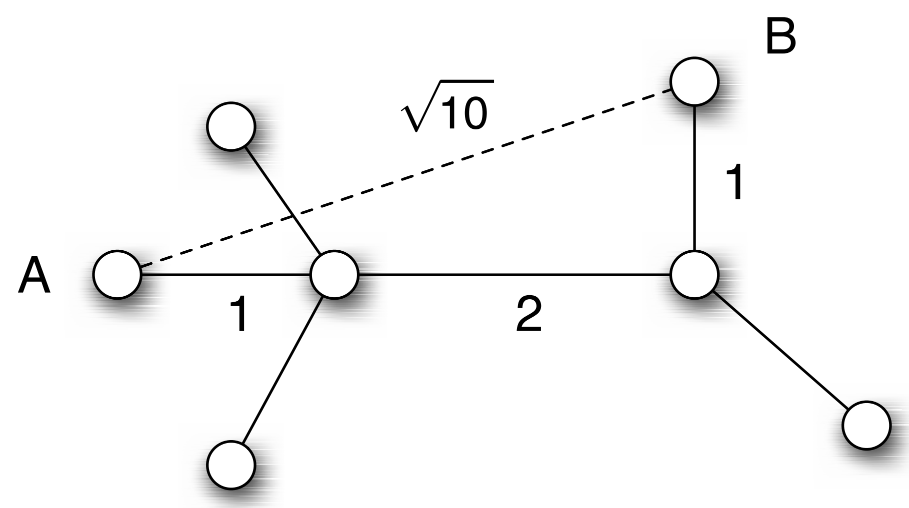
$$Q(u,v) = \frac{\Delta_{uv}}{\ell_{uv}}$$

$\langle Q(u) \rangle$ **Node Accessibility**: Average route factor from a node to all others:

$$\langle Q(u) \rangle = \frac{1}{N-1} \sum_{v} Q(u,v)$$

$\langle Q \rangle$ **Accessibility**: Average route factor for the whole network:

$$\langle Q \rangle = \frac{1}{N(N-1)} \sum_{u \neq v} Q(u,v)$$



Simple models of spatial networks

Random geometric graphs

General definition:

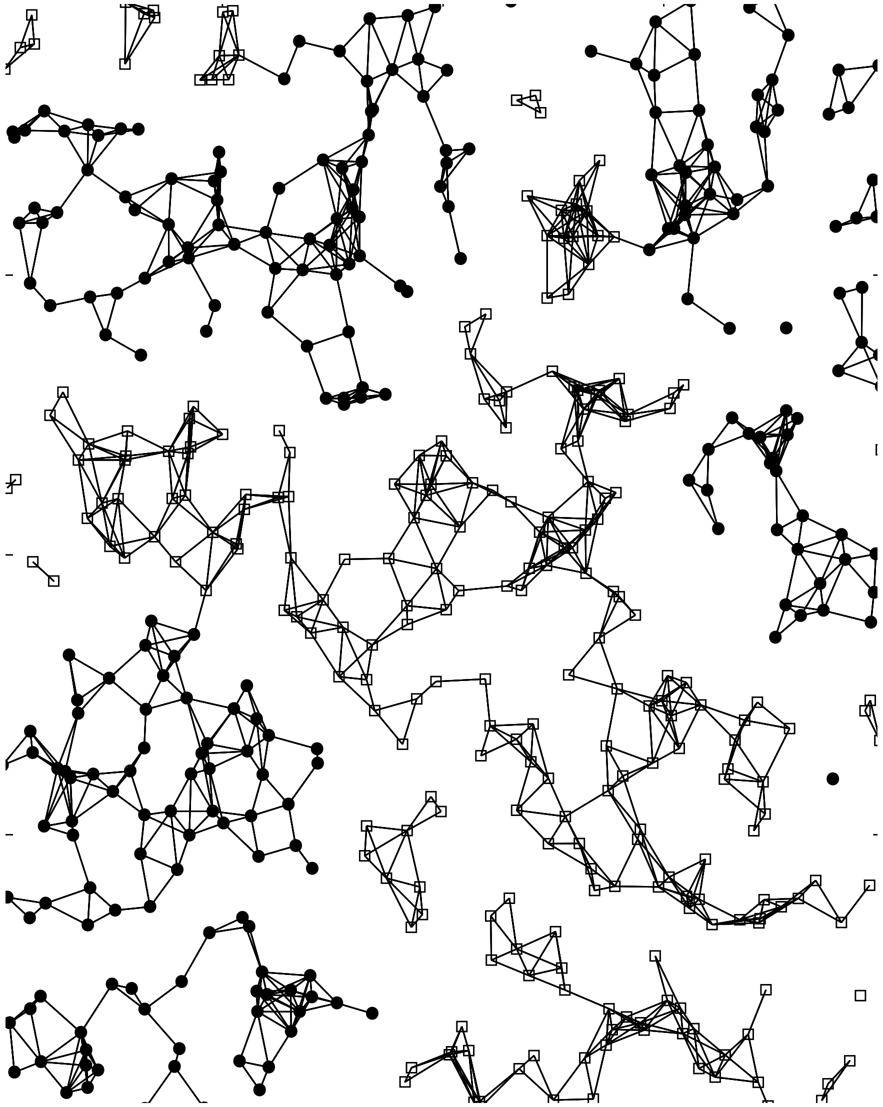
- Take a space and distribute nodes randomly
- Nodes are small spheres with radius r
- Two nodes are connected if their spheres overlap — separated with distance smaller than 2r
- Also called: disk-percolation

Degree distribution — Poisson distribution

Clustering coefficient (d=dimensions)

$$\langle C_d \rangle \sim 3\sqrt{\frac{2}{\pi d}}\left(\frac{3}{4}\right)^{\frac{d+1}{2}}$$

Independent of N contrary to random networks



Jesper Dall and Michael Christensen. “Random geometric graphs”. In: *Physical review E* 66.1 (2002), p. 016121.

Soft RGG

Soft RGG (Waxman random graph)

Soft RGG, or **Waxman Random Graphs**[a], starts as the RGG by distributing nodes at random in a space, but instead of adding links between all nodes closer than a certain distance, it assign edges between nodes according to a **deterrence function** f, i.e., a function defining how distance affects the probability of observing edges between nodes.

The Soft RGG can model an ER random graph if f is constant function, $f(\Delta) = p$. It can model a classic RGG if f is a threshold function with:

$$f(d) = \begin{cases} 1 & \Delta \leq r \\ 0 & \Delta > r \end{cases}$$

[a] Waxman 1988.

Deterrence function

Deterrence function

A deterrence function defines how the distance affects the probability of observing an edge. It can be a probability (bounded on $[0, 1]$), or define a change ratio.

1. It can be defined *a priori*, usually as a classic monotonically decreasing function, e.g., Negative exponential($f(\Delta) = e^{-\alpha\Delta}$) or Negative power ($f(\Delta) = \Delta^{-\alpha}$), with α a parameter. A typical example of negative power in geographical data is when the probability of observing an edge decreases as the square of the distance, i.e., $f(\Delta) = \frac{1}{\Delta^2}$
2. It can also be learned from data, either by fitting parameters of a predefined function (e.g., the α parameter above), or by using an *Ad-Hoc deterrence function*.

The gravity law

Formal description

Origin-destination matrix

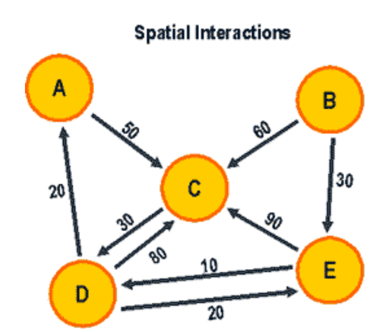
- Describe flow of individuals between locations
- Used since decades by geographers
- Definition:
    - divide the area of interest into zones (cells) labelled by $i=1 \ldots N$
    - count the number of individuals going from location i to location j

- directed
- weighted
- Beware:
    - strongly depends on the zone definition

O/D Matrix

$T(i,j)=$

	A	B	C	D	E	Ti
A	0	0	50	0	0	50
B	0	0	60	0	30	90
C	0	0	0	30	0	30
D	20	0	80	0	20	120
E	0	0	90	10	0	100
Tj	20	0	280	40	50	390



The gravity law

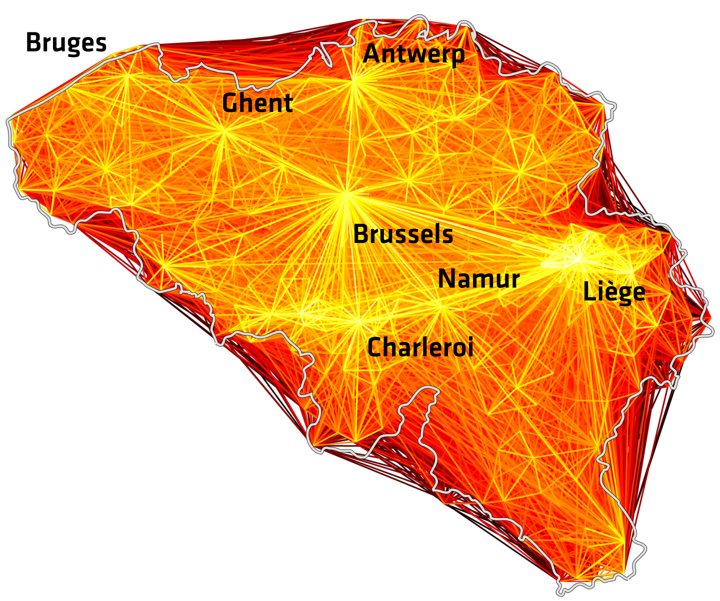
Number of trips from location i to location j is scaling as

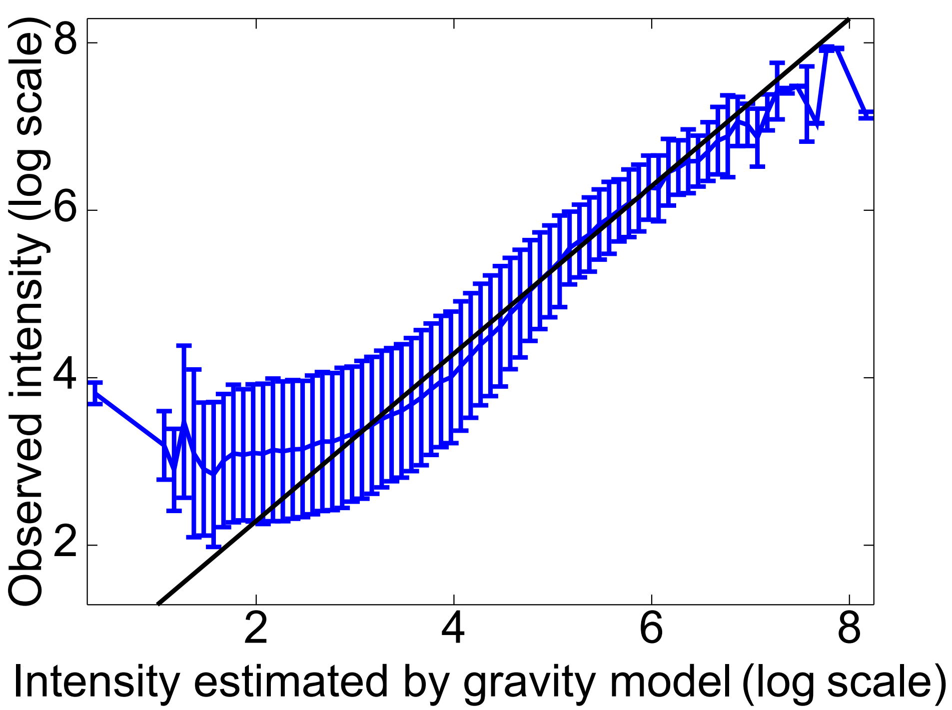
$$T_{ij} = K \frac{P_i P_j}{d_{ij}^{\sigma}}$$

- where $d_{ij} = d_E(i, j)$ is the distance between i and j
- $P_{i(j)}$ is the *population size* at location $I(j)$
- σ a parameter chosen or learned from data

Inter-city phone communication (Krings et.al.)

- mobile call communication intensity between Belgian cities





The gravity law - empirical summary

Both exponential and power-law dependence is observable

Network [Ref.]	N	Gravity law form	Results
Railway express [164]	13	P_iP_j/d_{ij}^{σ}	$\sigma = 1.0$
Korean highways [161]	238	P_iP_j/d_{ij}^{σ}	$\sigma = 2.0$
Global cargo ship [104]	951	$O_iI_jd_{ij}^{-\sigma}\exp(-d_{ij}/\kappa)$	$\sigma = 0.59$
Commuters (worldwide) [162]	n/a	$P_i^{\alpha}P_j^{\gamma}\exp(-d_{ij}/\kappa)$	$(\alpha, \gamma) = (0.46, 0.64)$ for $d < 300$ km
			$(\alpha, \gamma) = (0.35, 0.37)$ for $d > 300$ km
US commuters by county [163]	3109	$P_i^{\alpha}P_j^{\gamma}/d_{ij}^{\sigma}$	$(\alpha, \gamma, \sigma) = (0.30, 0.64, 3.05)$ for $d < 119$ km
			$(\alpha, \gamma, \sigma) = (0.24, 0.14, 0.29)$ for $d > 119$ km
Telecommunication flow [134]	571	$P_iP_jd_{ij}^{-\sigma}$	$\sigma = 2.0$

The gravity law

Number of trips from location i to location j is scaling as

$$T_{ij} = K\frac{P_i P_j}{d_{ij}^{\sigma}}$$

- where $d_{ij} = d_E(i,j)$ is the distance between i and j
- $P_{i(j)}$ is the *population size* at location $i(j)$

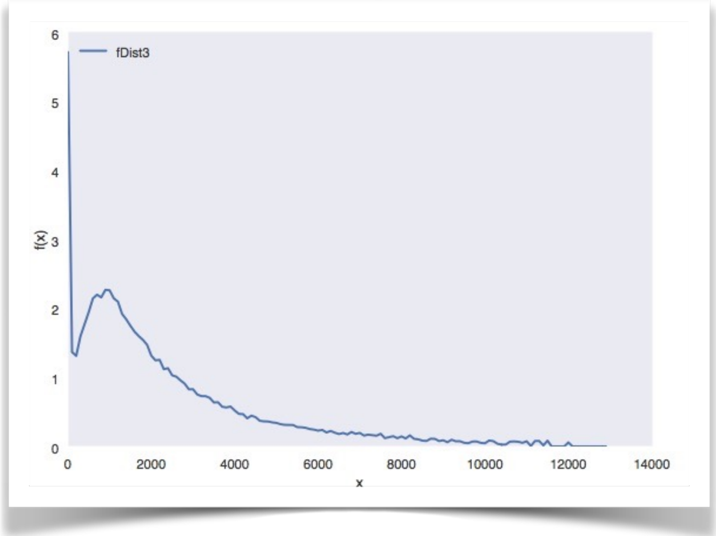
- In a general form: $T_{ij} \sim P_i P_j f(d(i,j))$
  - where $f(d(i,j))$ is the deterrence function describing the effect of space

Ad-hoc deterrence function

Agnostic deterrence function

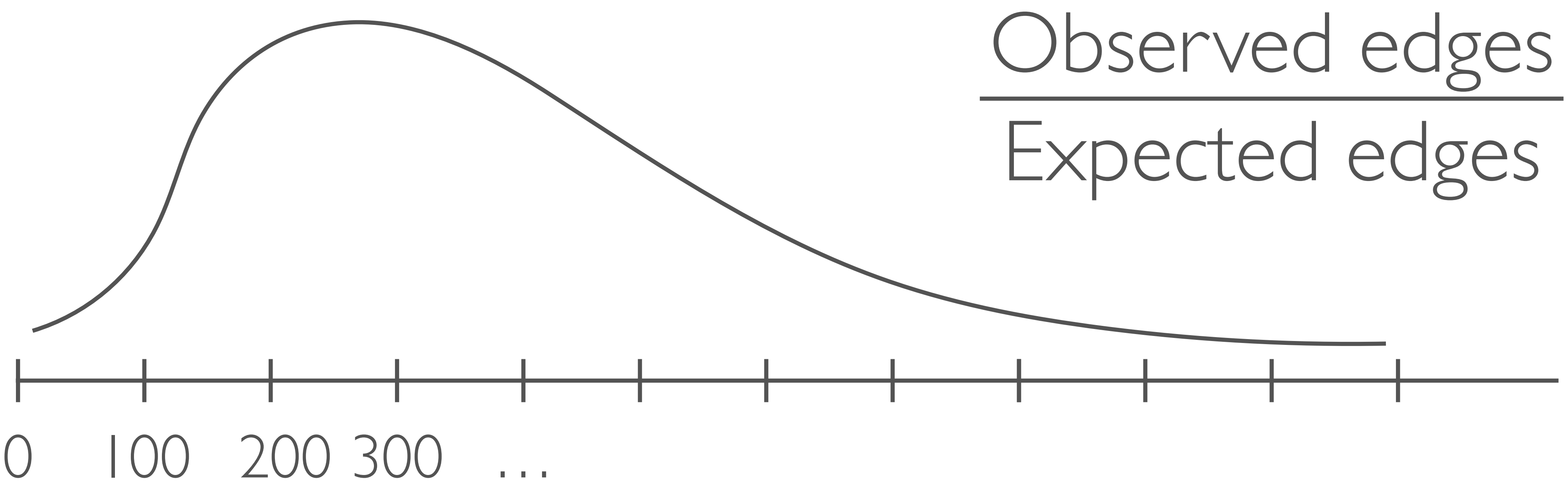
- The influence of distance might be more complex than a power-law or an exponential. In particular, it is often non-monotonic (first increasing, then decreasing. Think of airplanes, bicyles, public transports… unlikely to use for short distances)
- A deterrence function can be learned from data
- Computed by comparing the number of trips observed at a given distance with the number of trip expected if distance has no effect (a configuration model)

$f(d)$



Distance d

Ad-hoc deterrence function



$$f(d) = \frac{\sum_{i,j|\Delta_{ij}=d} A_{ij}}{\sum_{i,j|\Delta_{ij}=d} M_{ij}}$$

with A_{ij} the adjacency matrix (or weight matrix) of the observed graph and M_{ij} the probability of observing an edge (or weight of edges) between nodes i and j according to the chosen null model. For instance, with the simplest hypothesis that edges occur completely at random, $\forall_{i,j}, M_{ij} = d$.

The gravity law - as a network null model

Usage as a **network** null model

- Consider a spatial network (e.g., phone calls, trips, etc.)

- Fit a gravity model best explaining the observed network. If the population is unknown or not relevant, the degrees of nodes (in/out degrees in directed networks) can be used as a "*population*"

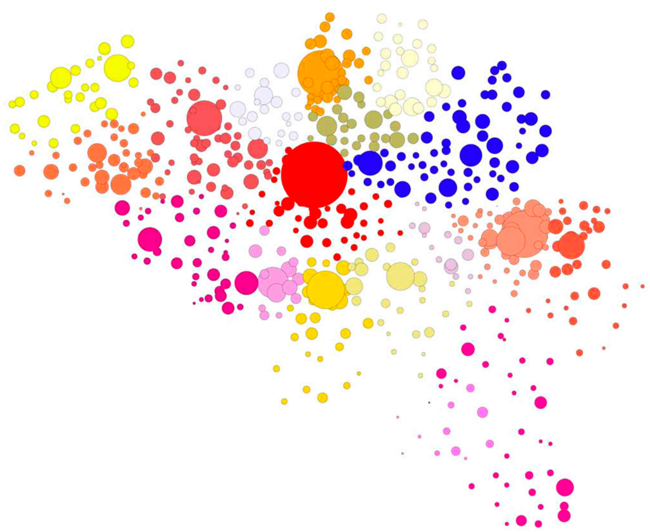
- =>Random model with a given edge probability for each pair of node

- The obtained network is a null model to which the observed network can be compared

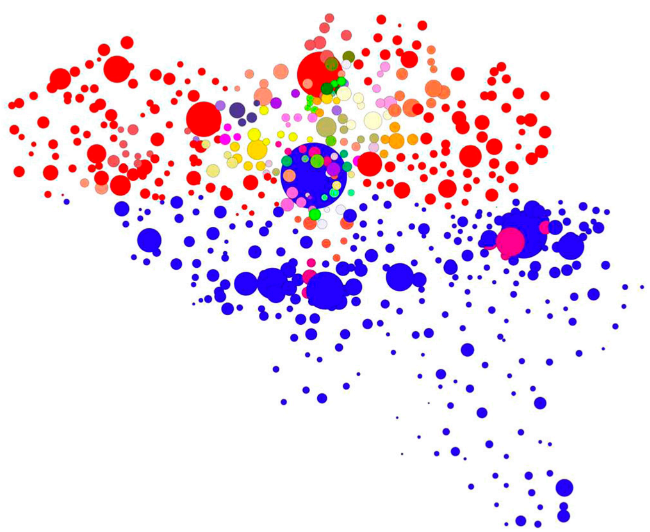
The gravity law - as a network null model

Example of application: Space-independent communities

- In the usual modularity, the fraction of internal edges is compared between the observed network and a configuration model.
- One can replace the configuration model by a gravity model



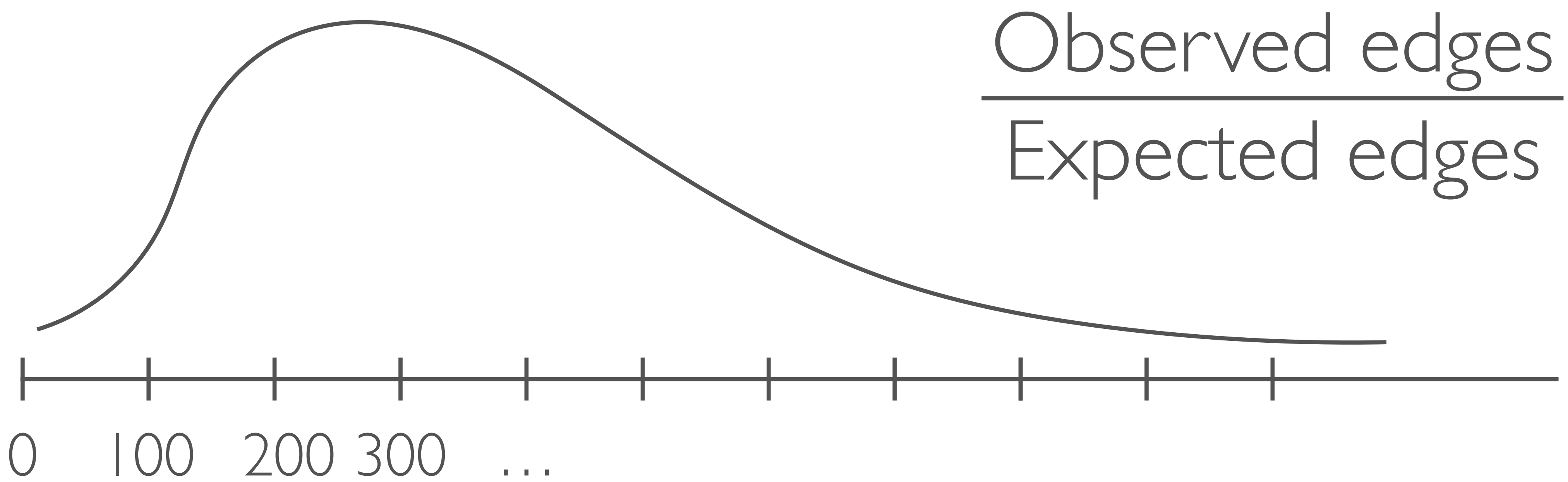
Space-dependent communities



Space-independent communities

Expert, P., Evans, T. S., Blondel, V. D., & Lambiotte, R. (2011). Uncovering space-independent communities in spatial networks. *Proceedings of the National Academy of Sciences*, *108*(19), 7663-7668.

Deterrence function and gravity model



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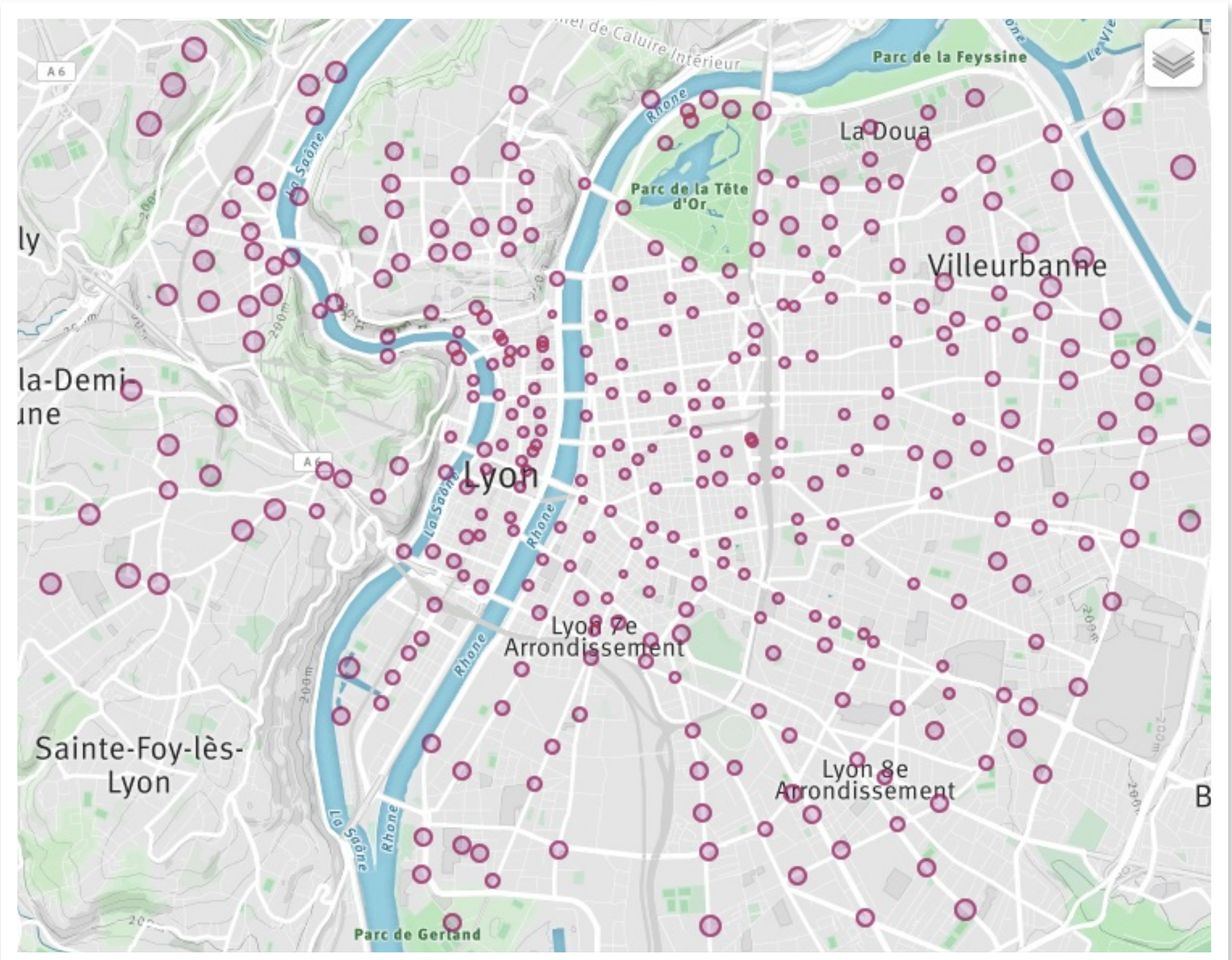
$$f(d) = \frac{\sum_{i,j|d_{ij}=d} A_{ij}}{\sum_{i,j|d_{ij}=d} \frac{k_i k_j}{2L}}$$

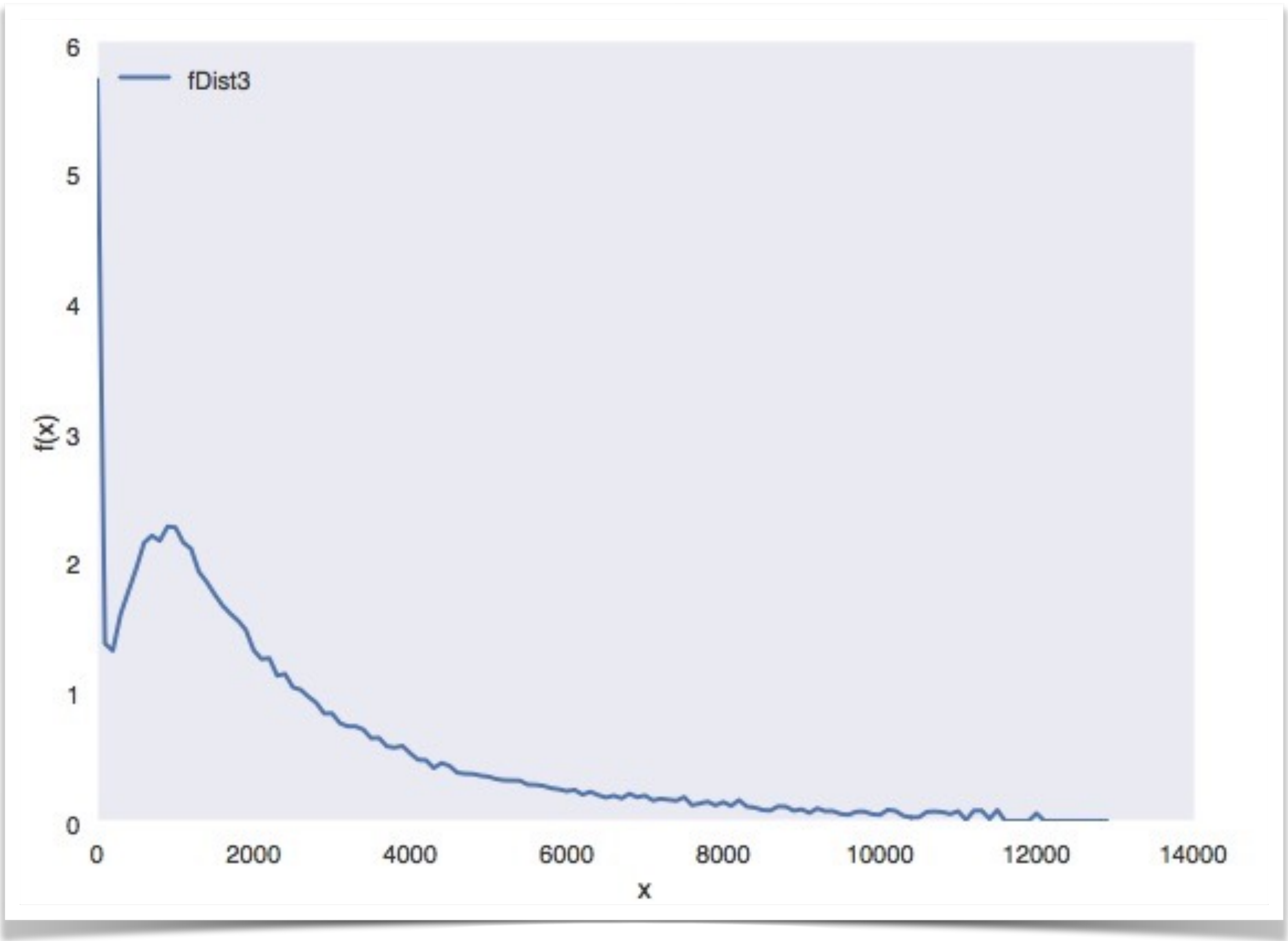
The gravity law - example



Nodes: Vélo'v station (2D position)
Edges: number of trips over a period

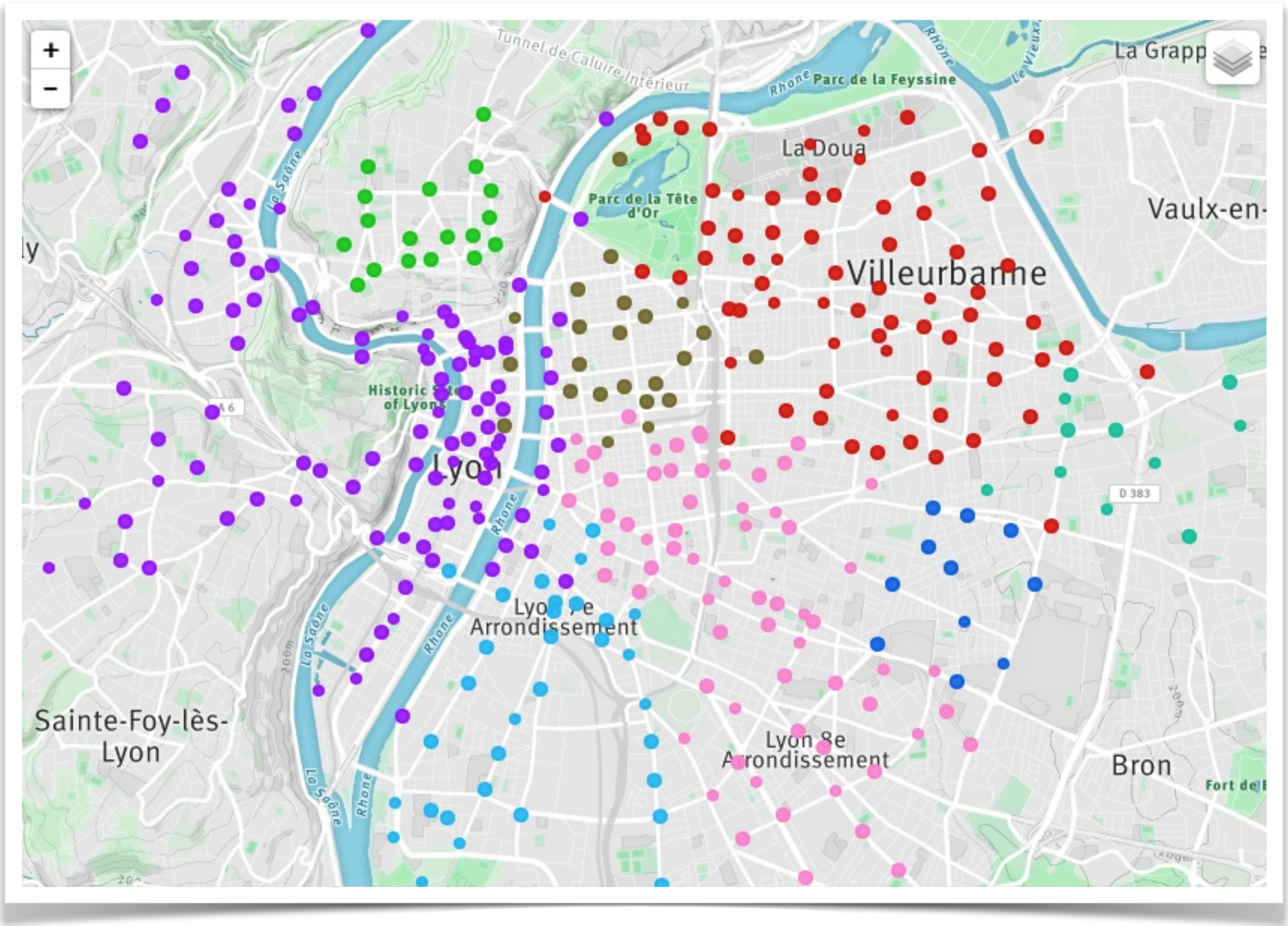
The gravity law - example

$f(d)$



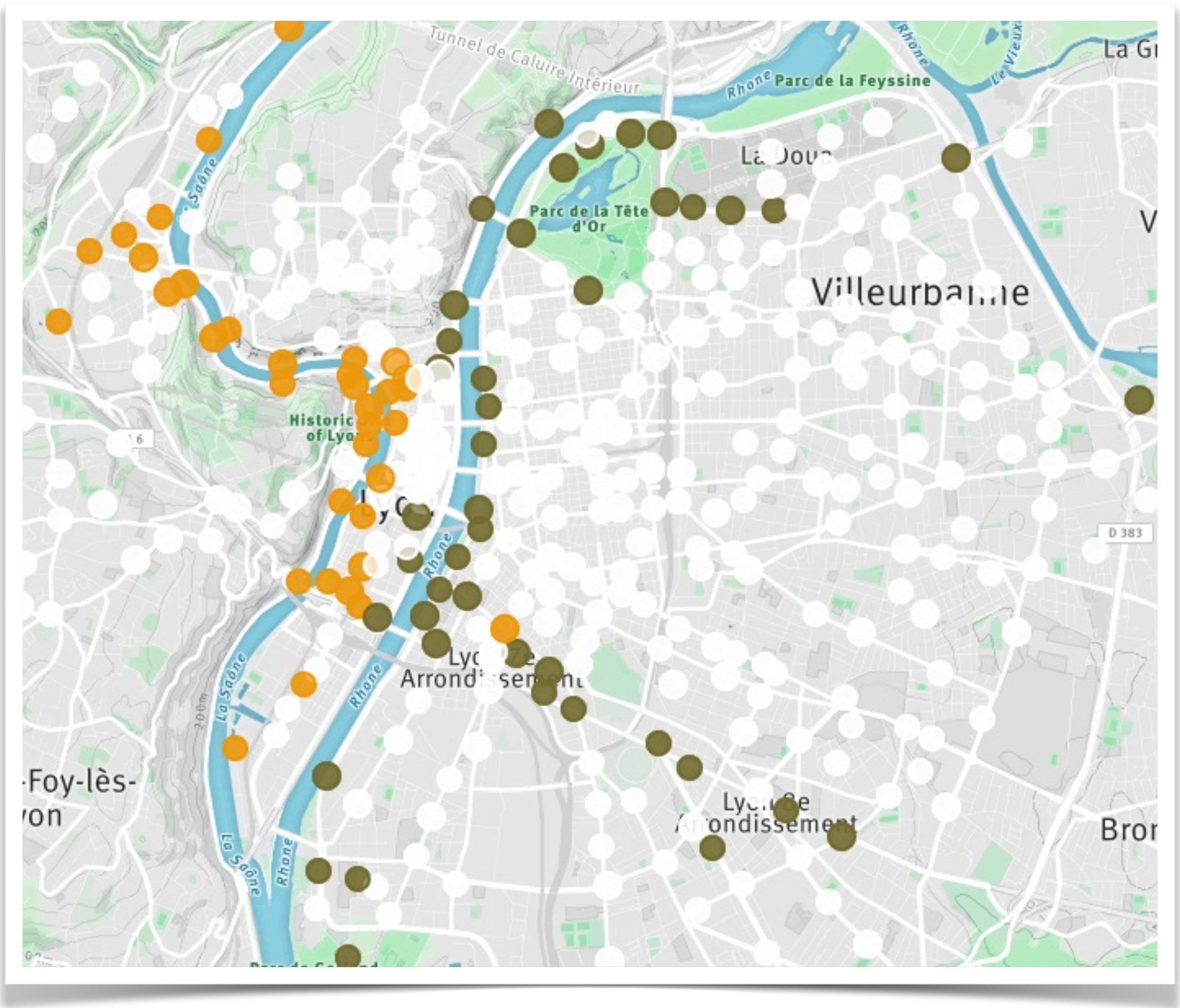
Distance *d* *(meters)*

The gravity law - example



Space-dependent communities

The gravity law - example



Some (social) space-independent communities that were previously *hidden* by spatial constraints

The radiation law

The radiation law

Limitations of the gravity law

1. Requires previous data to fit

2. The number of travelers between destinations depends only on their populations and distances. In reality, this value depends probably of other *opportunities*

The radiation law

Intuition: Model how people move for jobs

1. Individuals look for job in all cities

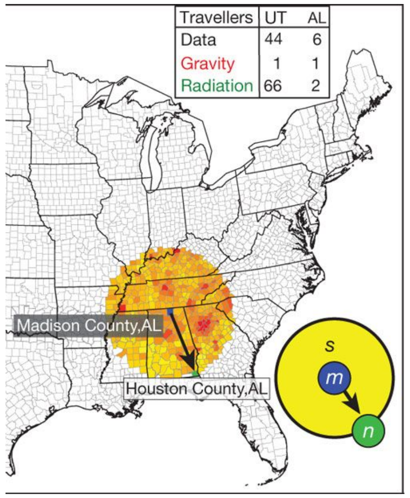
2. Each city has a number of job opportunities

    - Each job has a value of *interest*, considered random

3. What is the probability for a job-seeker to choose a job in city *c* located at distance *d?*

    - Depends only on how many jobs offered in cities at a distance equal or lower than *d* (probability to find a better job closer)

The model is parameter-free!



The radiation law

The model can be formulated in terms of radiation and absorption

- take locations i and j with populations (in-degree) m_i and n_j and at distance r_{ij}
- denote s_{ij} the total population in the circle with radius r_{ij} centered at i (excluding the source and destination population)
- T_i is the number of commuters (out-degree)

Radiation Law of Spatial Interactions

The **Radiation Law**[a] is another random spatial model. Unlike previous ones, it does not depends on a deterrence function, and is parameter-free. It is based on the principle of relative opportunities: the probability of observing an interaction from i to j depends on P_i^{out}, P_j^{in}, and the sum of all P_k^{in} for $\Delta_{ik} < \Delta_{ik}$, i.e., other opportunities accessible at a shorter distance. More formally:

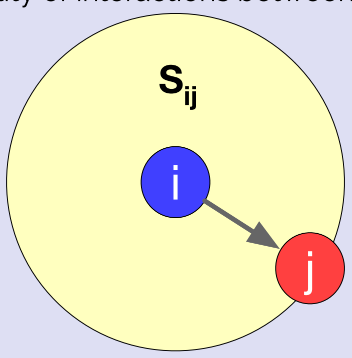
$$R_{ij} = k_i^{out} \frac{k_i^{out} k_i^{in}}{(k_i^{out} + s_{ij})(k_i^{out} + k_i^{in} + s_{ij})}$$

With $s_{ij} = \sum_{u \in V, \Delta_{iu} < \Delta_{ij}} k_u^{in}$ the sum of opportunities at a shorter distance than the target.

[a] Simini et al. 2012.

Radiation Law of Spatial Interactions

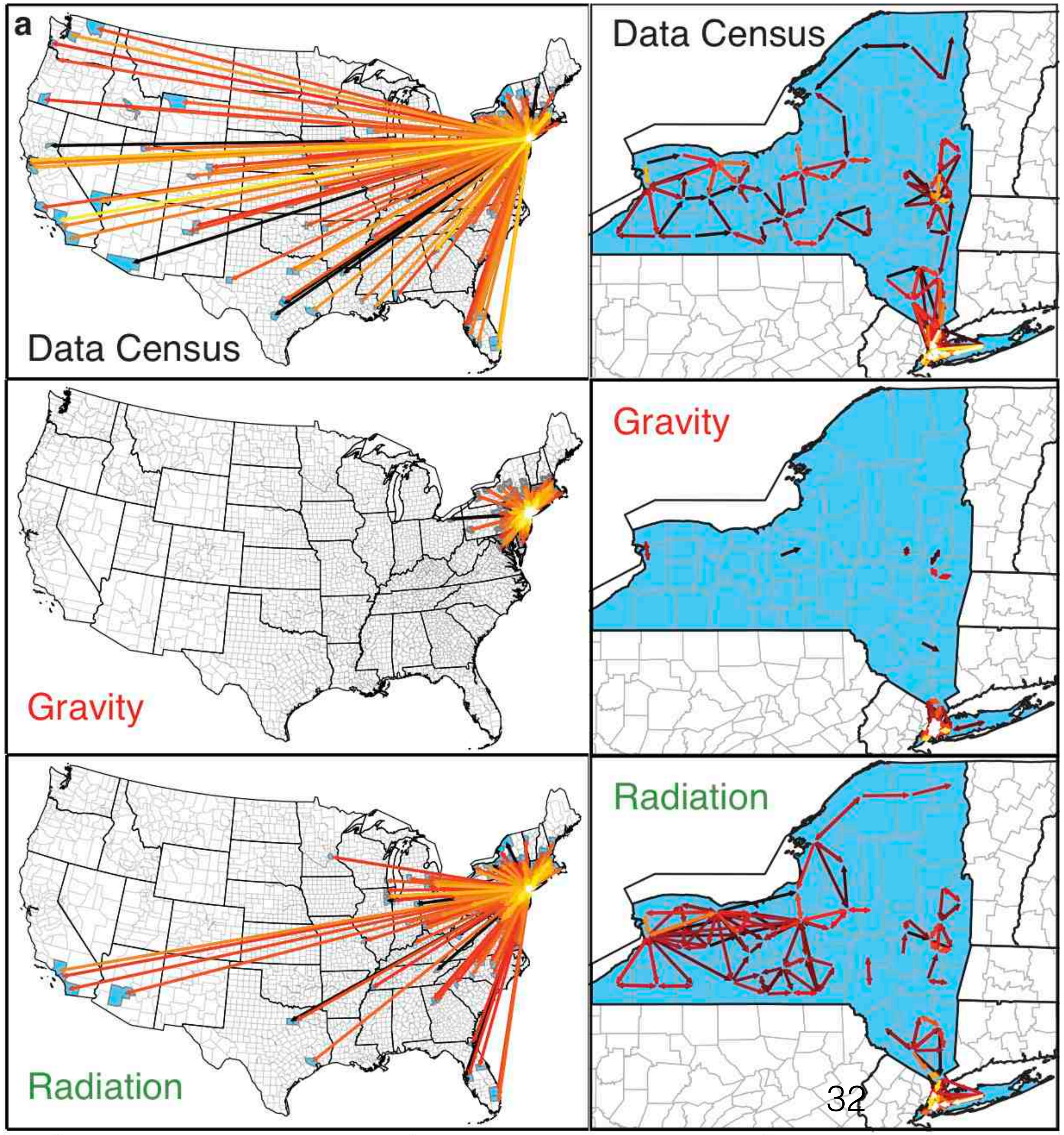
Illustration of the zone s_{ij} in which opportunities decrease the probability of interactions between i and j.

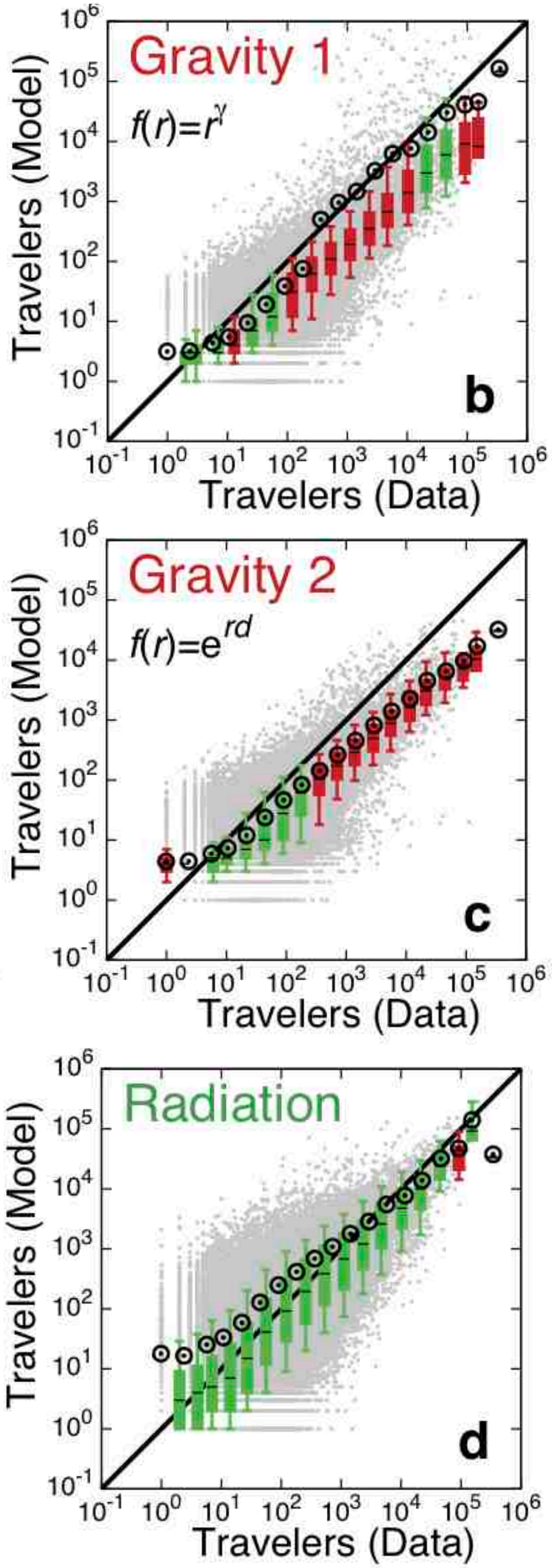


The radiation law

Comparison with census data and the gravity law predictions

Simini. et.al, Nature 2010





Radiation Law VS Gravity Law

+ Radiation:
    - No parameters
    - Two nodes of same degrees at similar distance can have different edge probability based on their location
+ Gravity:
    - Customizable deterrence function… The real world is complex !

MULTI-PARTITE GRAPHS

- Bi-partite: there exists 2 kinds of nodes, and links can only connect nodes of different types
  - Multi-partite: similar but with more than 2 types. (less common)
  - Not strictly different from normal graphs: if you don't know the two categories of nodes, it looks like any network

- Bi-partite networks are quite commonly use
  - Actors - Films
  - Clients - Products
  - Reserchers - conferences/institutions
  - …

MULTI-PARTITE GRAPHS

- The problem is that some definitions of normal graphs become meaningless
  - Clustering coefficient
  - Modularity
  - …

MULTI-PARTITE GRAPHS

Modularity: do not count pairs of nodes of same types

$$Q_{\mathrm{B}} = \frac{1}{m} \sum_{u=1}^{r} \sum_{v=1}^{c} (\tilde{A}_{uv} - P_{uv}) \delta(g_u, h_v) = \frac{1}{m} \sum_{u=1}^{r} \sum_{v=1}^{c} \left(\tilde{A}_{uv} - \frac{k_u d_v}{m} \right) \delta(g_u, h_v),$$

MULTI-PARTITE GRAPHS

Clustering Coefficient:

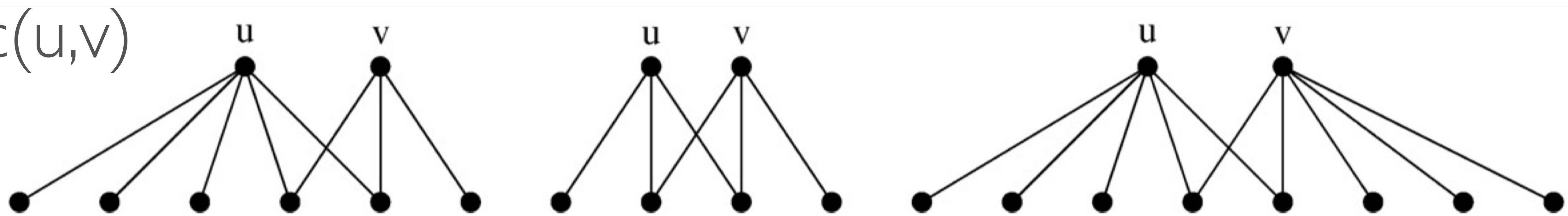
Of a pair

$$cc_{\bullet}(u, v) = \frac{|N(u) \cap N(v)|}{|N(u) \cup N(v)|}$$

Of a Node: Average among nodes N at distance 2

$$cc_{\bullet}(u) = \frac{\sum_{v \in N(N(u))} cc_{\bullet}(u, v)}{|N(N(u))|}$$

cc(u,v)



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MULTI-PARTITE GRAPHS

- Large literature on the topic, in particular applications to **recommendation**
  - Users - products => propose the right products to the right user

Kunegis, J., De Luca, E. W., & Albayrak, S. (2010, June). The link prediction problem in bipartite networks. In *International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems* (pp. 380-389). Springer, Berlin, Heidelberg.

Barber, M. J. (2007). Modularity and community detection in bipartite networks. *Physical Review E*, *76*(6), 066102.

Zhang, P., Wang, J., Li, X., Li, M., Di, Z., & Fan, Y. (2008). Clustering coefficient and community structure of bipartite networks. *Physica A: Statistical Mechanics and its Applications*, *387*(27), 6869-6875.

MULTI-PARTITE GRAPHS

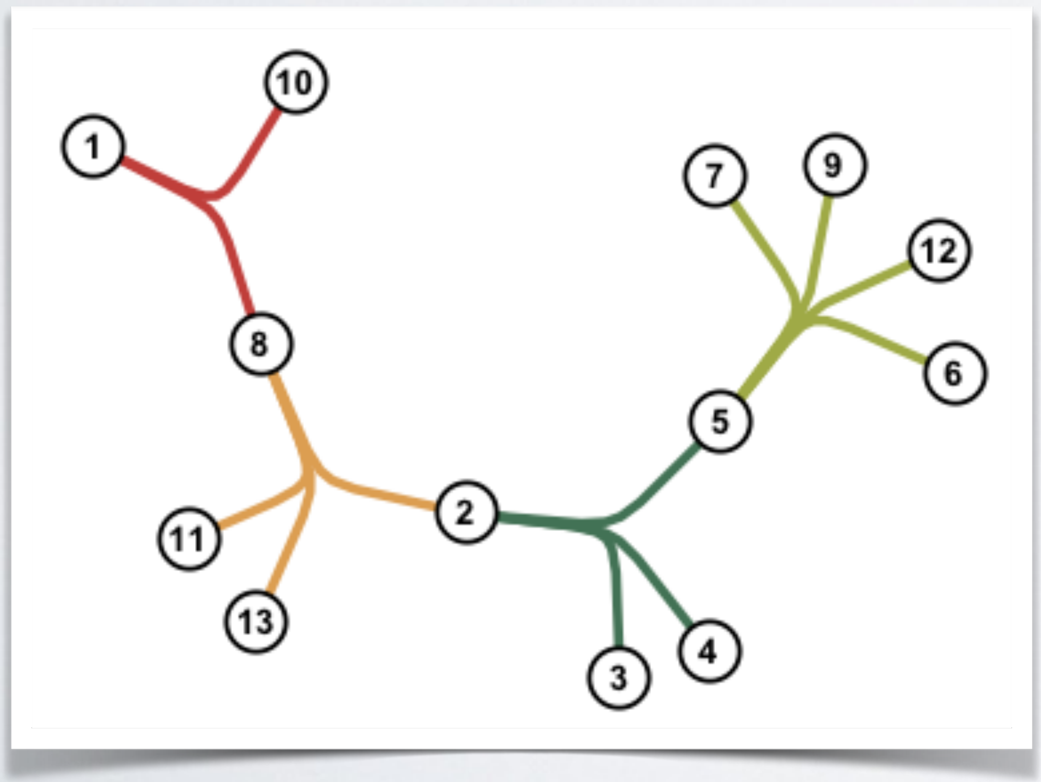
- A bipartite graph can be **projected** on one of its node-set
- One set of of nodes remain as nodes
- Those nodes are connected if they share a neighbor in the bipartite graphs
  - Variations: threshold, corrected by a null-model, etc.

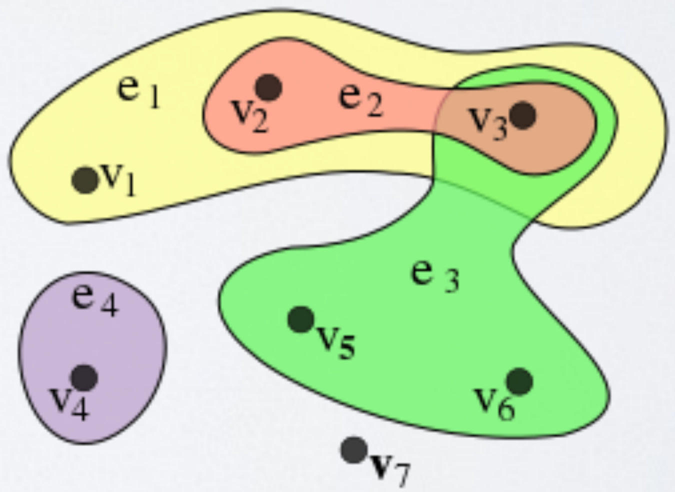
SELECTION OF INTERESTING NETWORK REPRESENTATIONS

HYPERGRAPHS

HYPERGRAPHS

- "Generalization" of graph
- An edge is not limited to 2 extremities





HYPERGRAPHS

- Most common usage: represent a single event involving several nodes

- In social networks: 10 students attending a same course A
  - Normal network: 45 undirected edges. Giant clique. Very dense
    - Problem: if 5 attend another course B and the others another course C => no way to see who worked with whom (a single clique, with double links or weights=2)
  - Hypergraph: A single link with ten endpoints
    - And we can add 2 single links with 5 endpoints and still differentiate attendances

- Another example: in Bitcoin, transactions are multi-input, multi-output. Some transactions have 1000 input, 1000 output
  - 500 000 links for a single transaction in a normal network!

HYPERGRAPHS

- In practice, very few direct usages
  - Too difficult to handle ? Too different from normal networks?
- Hypergraphs can be transformed in bi-partite graphs
  - Social Network: student nodes and class nodes
  - Bitcoin: transaction nodes and address nodes

MULTILAYER NETWORKS

MULTILAYER NETWORKS

- Multiplex network
- Multislice network
- Multitype network
- Heterogenous information network

[Kivela 2014]

MULTILAYER NETWORKS

- Can be used to represent:
    - Several types of relations between the same nodes
        - Bus transportation network
        - Bicycle transportation network
        - Car transportation network
        - …

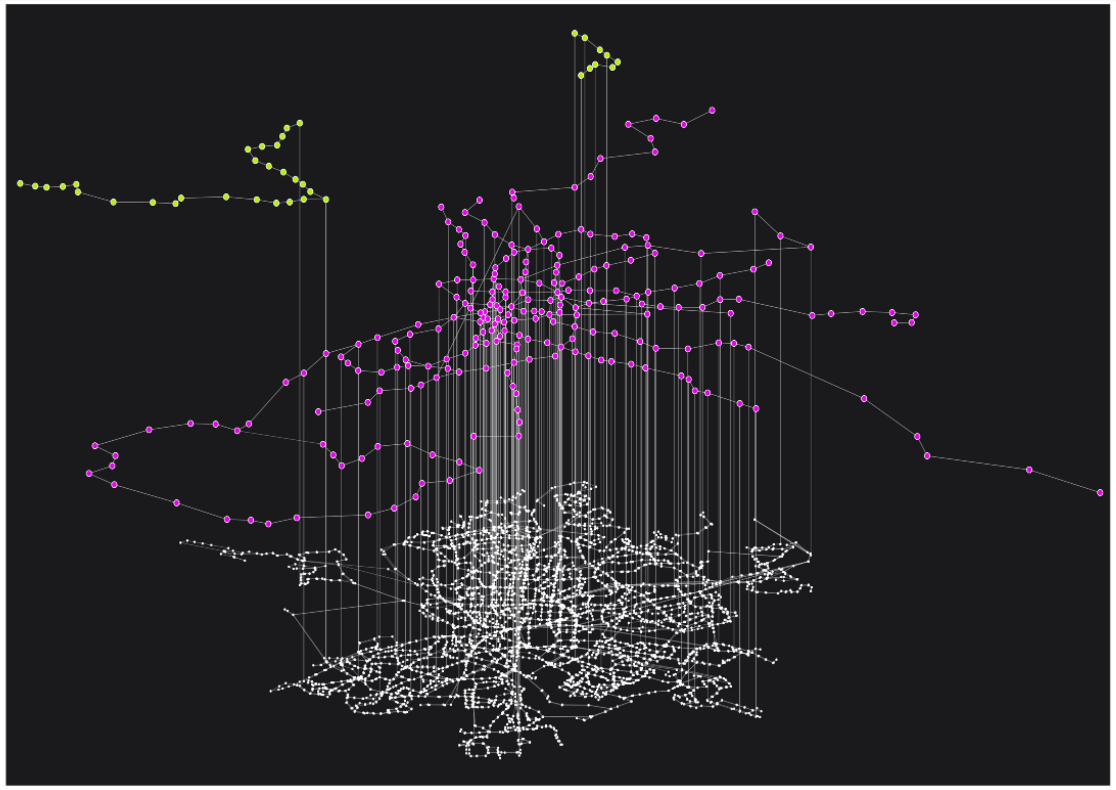
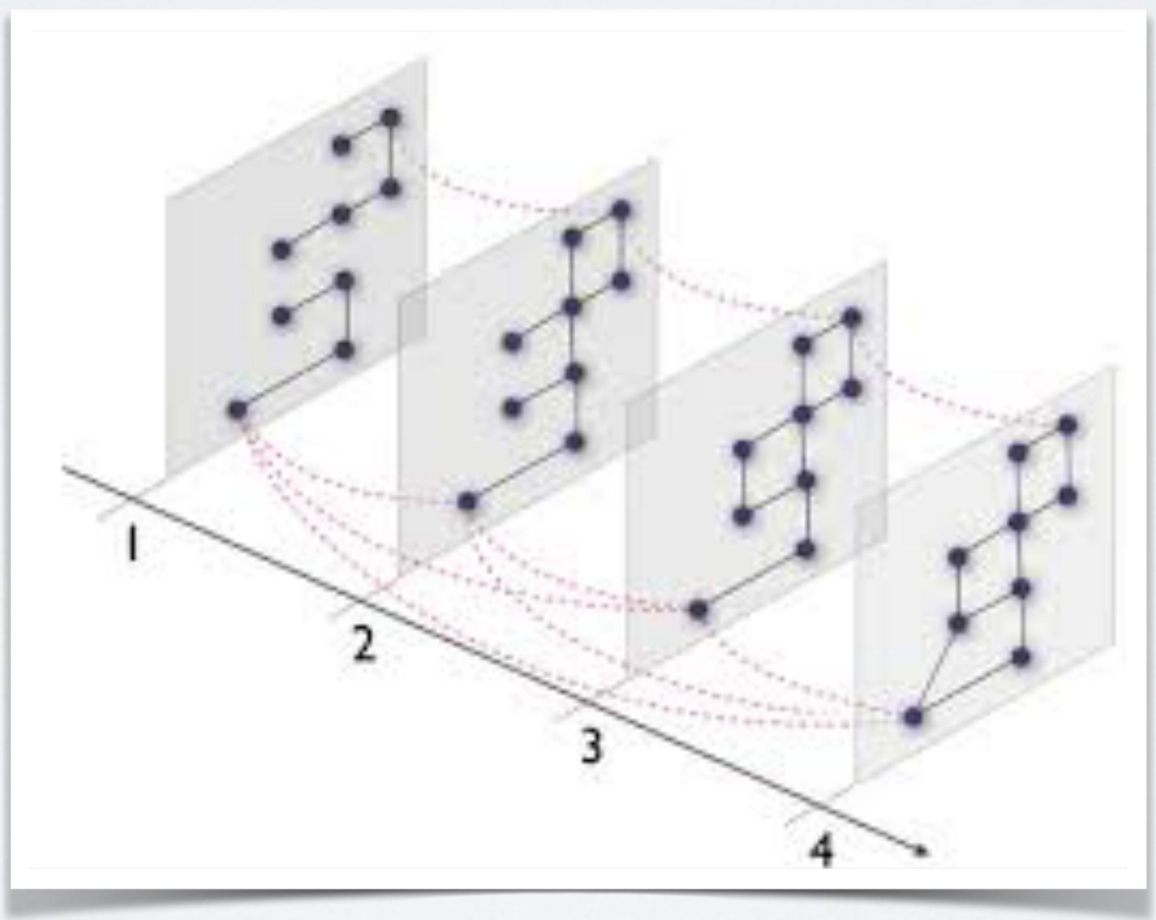


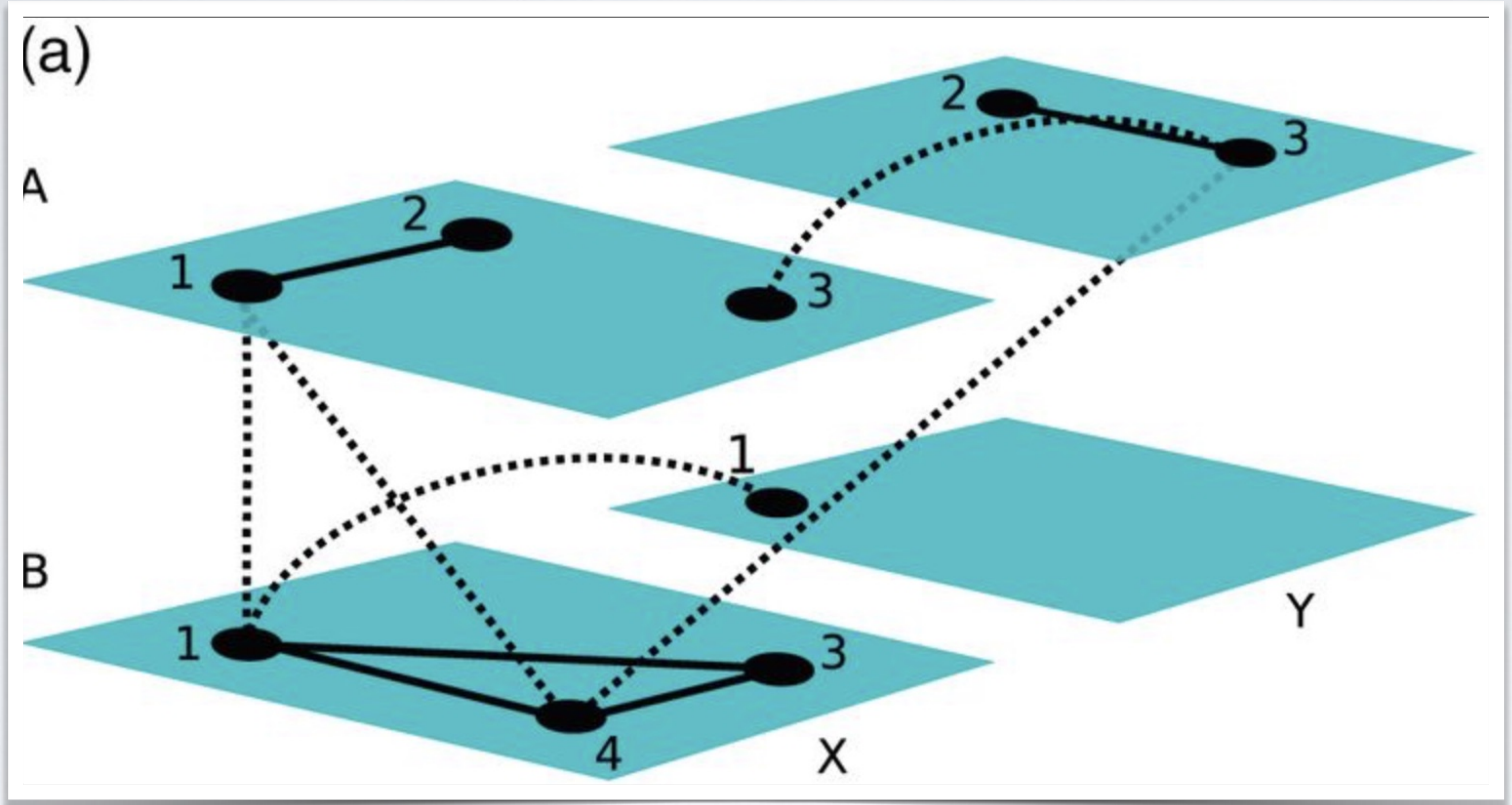
Figure 2. Superlayer representation of the Madrid transportation system. The figure represents the three transportation modes considered: tram (yellow nodes, upper layer), metro (purple nodes, mid layer) and buses (white nodes, bottom layer). See Table1 for statistics of these layers.

MULTILAYER NETWORKS

- Can be used to represent:
  - Several snapshots of the same network



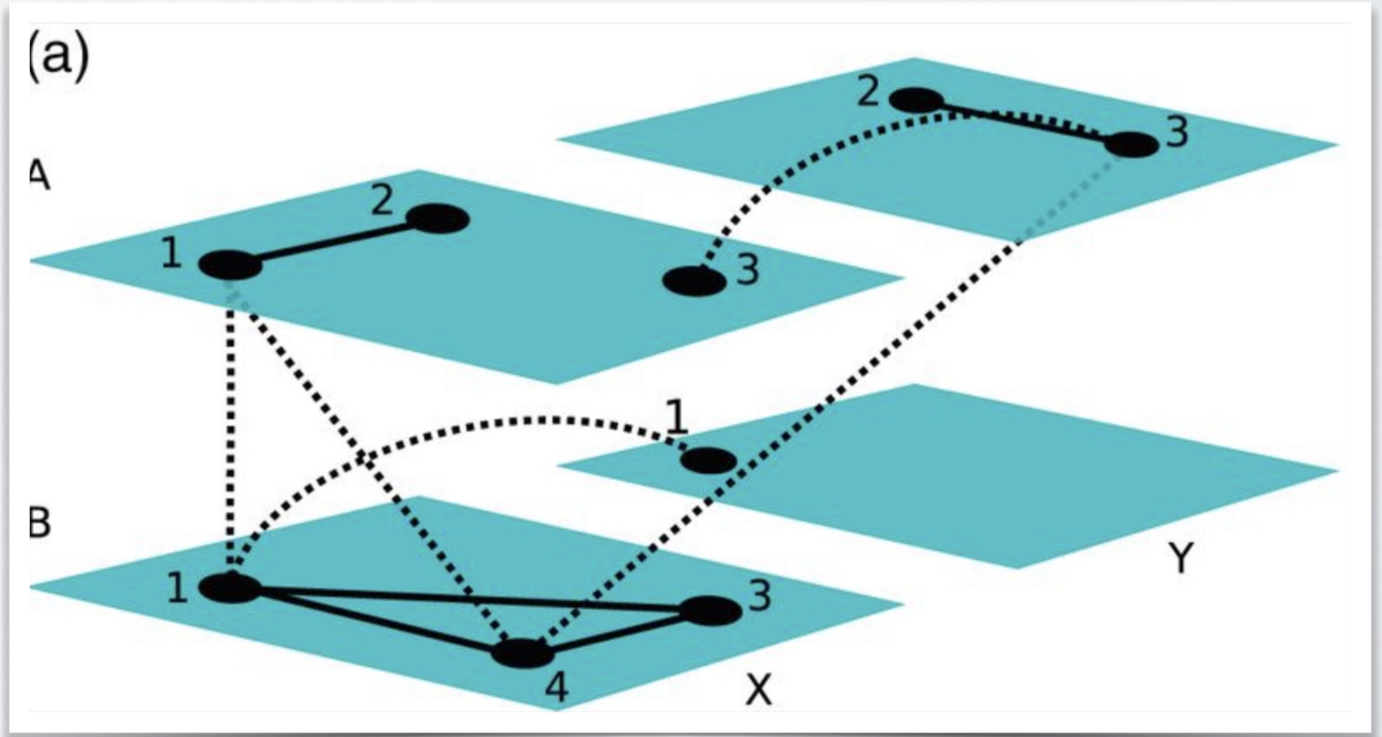
MULTILAYER NETWORKS



Both/Other

MULTILAYER NETWORKS

- Relations can be:
  - Only between **same** nodes in different layers
    - Public transport interconnection
  - Between different nodes in different layers
    - Information transfert form person A on Facebook to person B on Instagram.

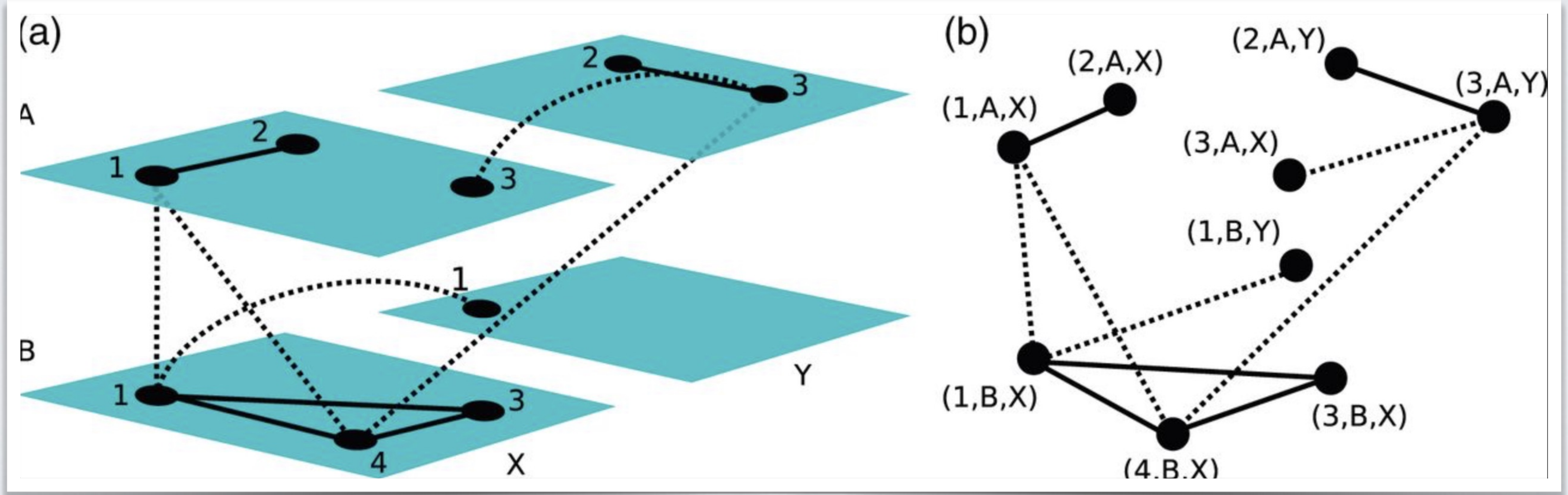


MULTILAYER NETWORKS

- All usual definitions on static networks can be extended to multilayer networks
  - Degree, clustering coefficient, community detection…
- The problem is that there are many ways to do it, and it depends on what your layers represent
  - Degree of a person on a multilayer network of facebook, Twitter, Linked-in?
- If you used a multilayer network, it is because it was not well summarized by a single network…

MULTILAYER NETWORKS

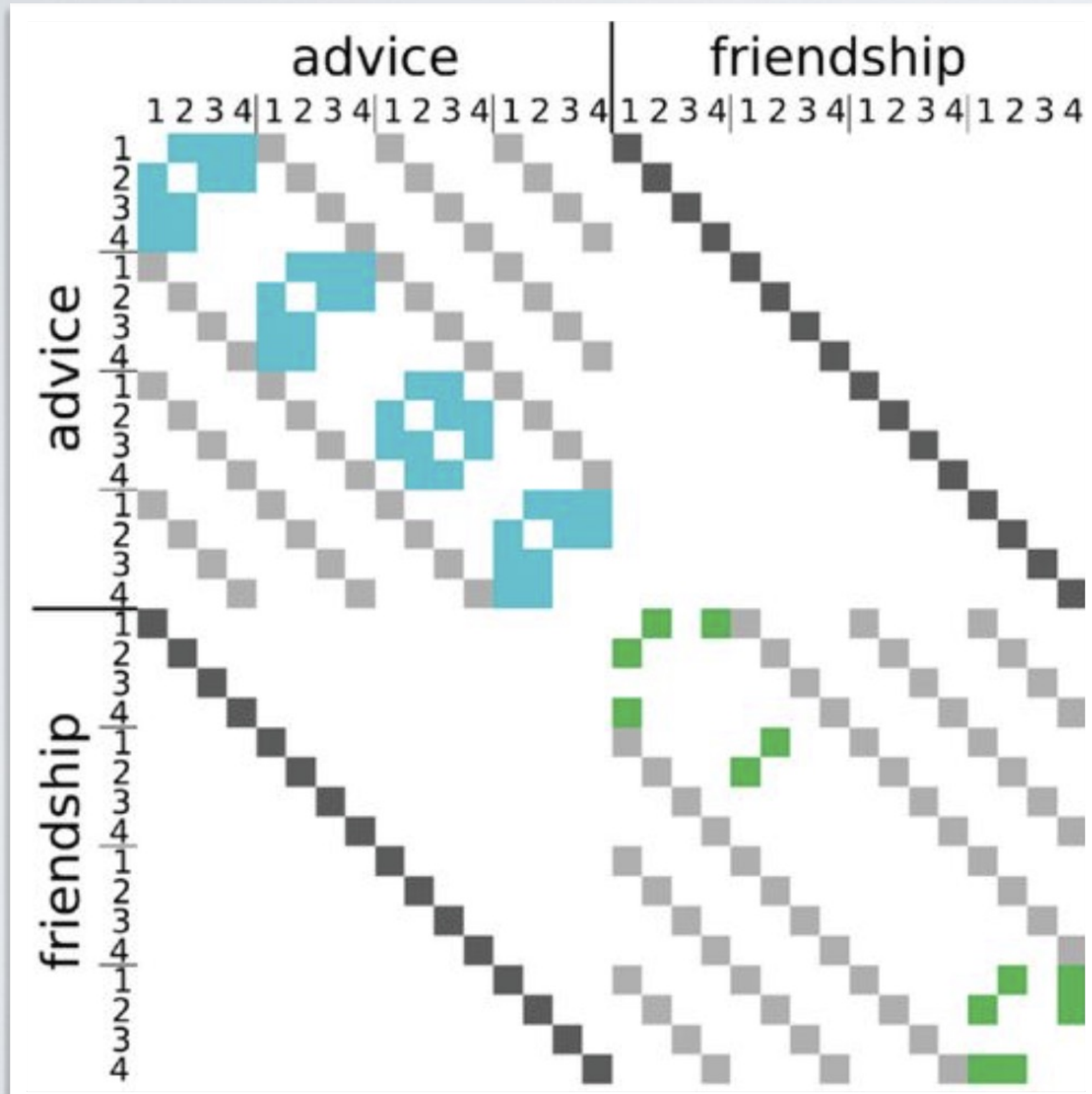
A simple idea: multilayers networks can be transformed into simple networks



MULTILAYER NETWORKS

- Matrix representation:
  - Many algorithms on networks work on adjacency matrices
- Solution: Supra-adjacency matrix
  - Or flattened tensors

MULTILAYER NETWORKS



Blue, green: intra-layer

gray: inter-layer 1

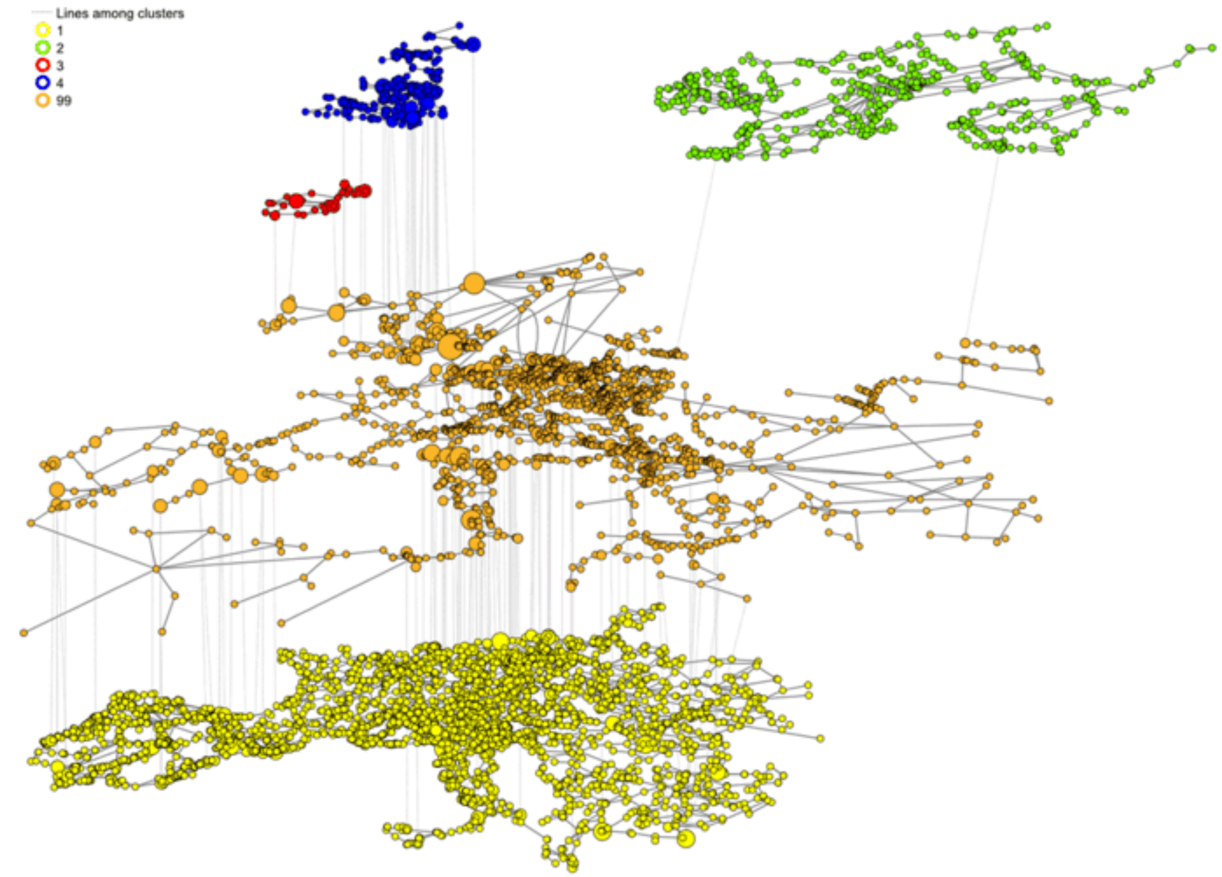
black: inter-layer2

Cognitive map: relations between
4 people seen by each of these 4 people

Interdependent networks

Interdependent networks:

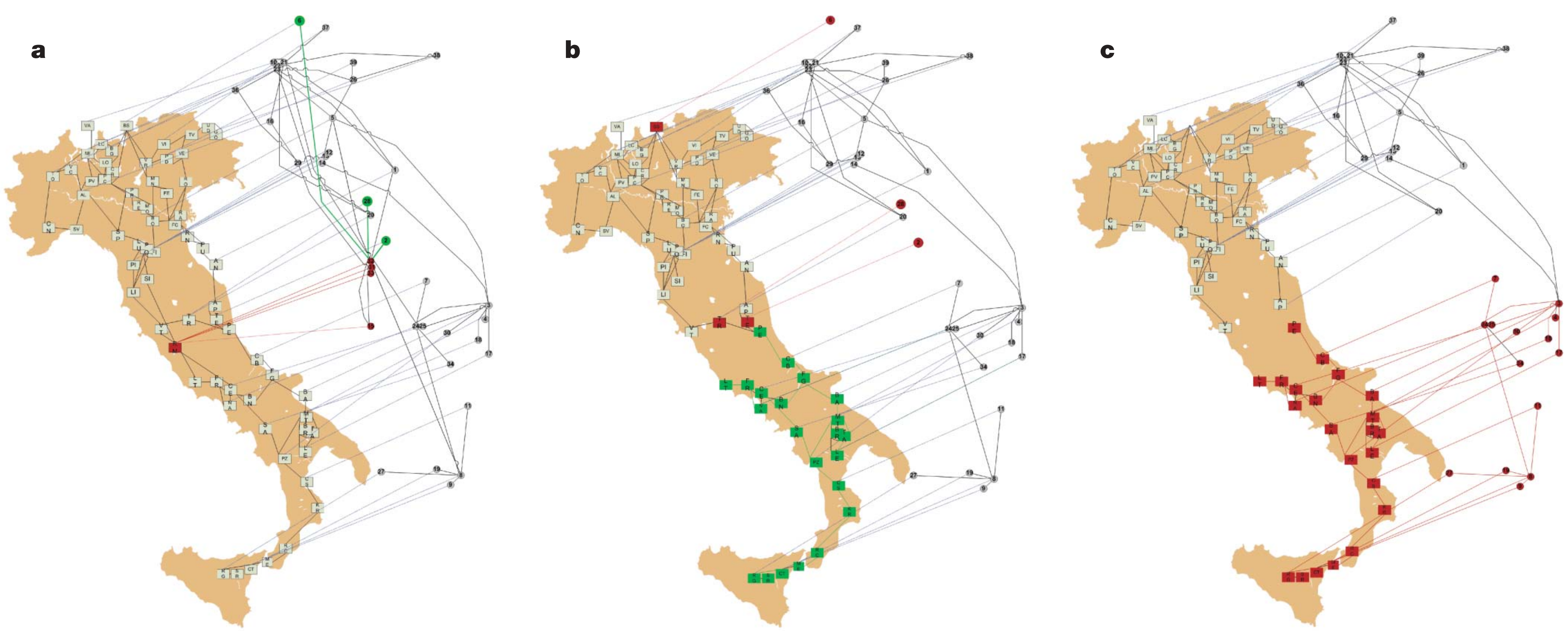
- links between networks assign the dependency between nodes in different layers
- The identities of nodes are not necessarily the same in different layers



Infrastructure networks

Example: 2003 Italy blackout

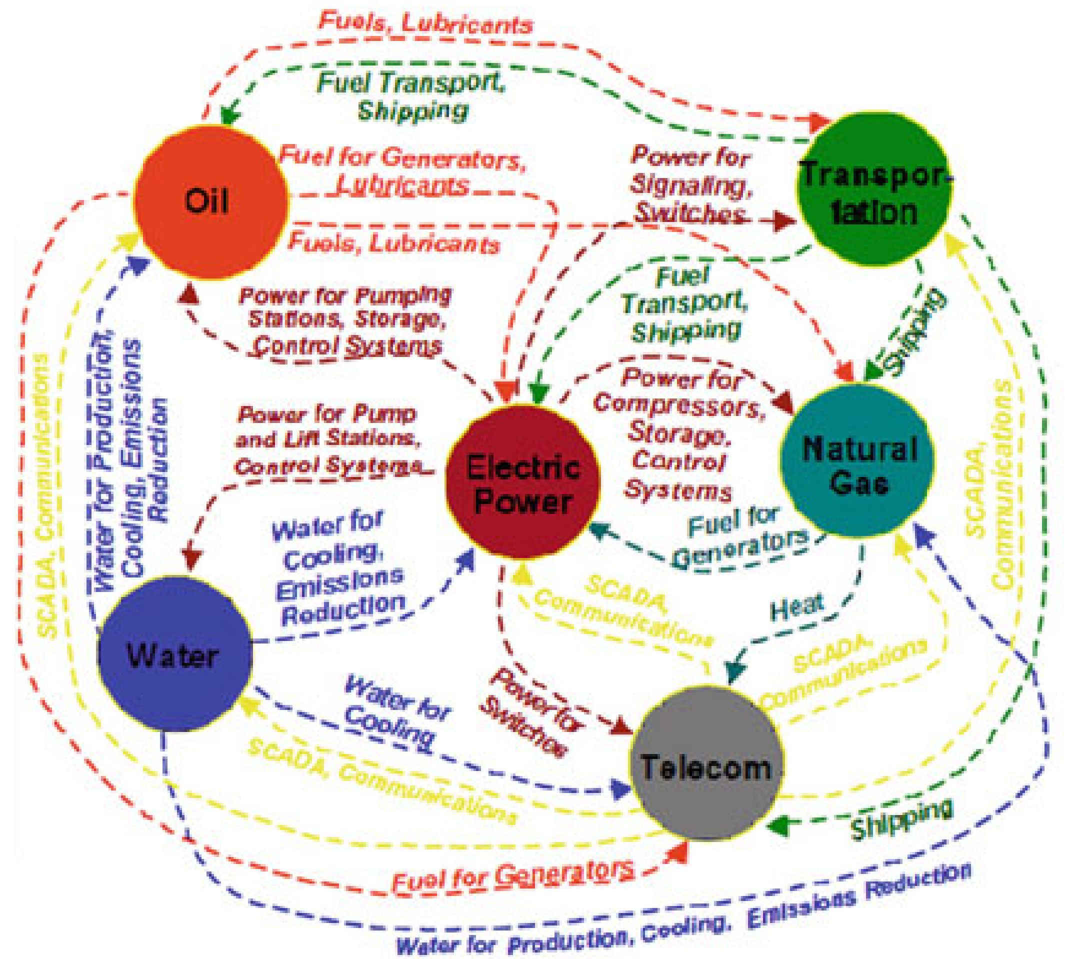
- A power line between Italy and Switzerland was damaged by storm
- Power outage for 12 hours in Italy and spread to Switzerland for 3 hours
- 56 millions of people without electricity
- 110 trains cancelled, All flights were cancelled



Infrastructure networks

Interdependent infrastructure networks

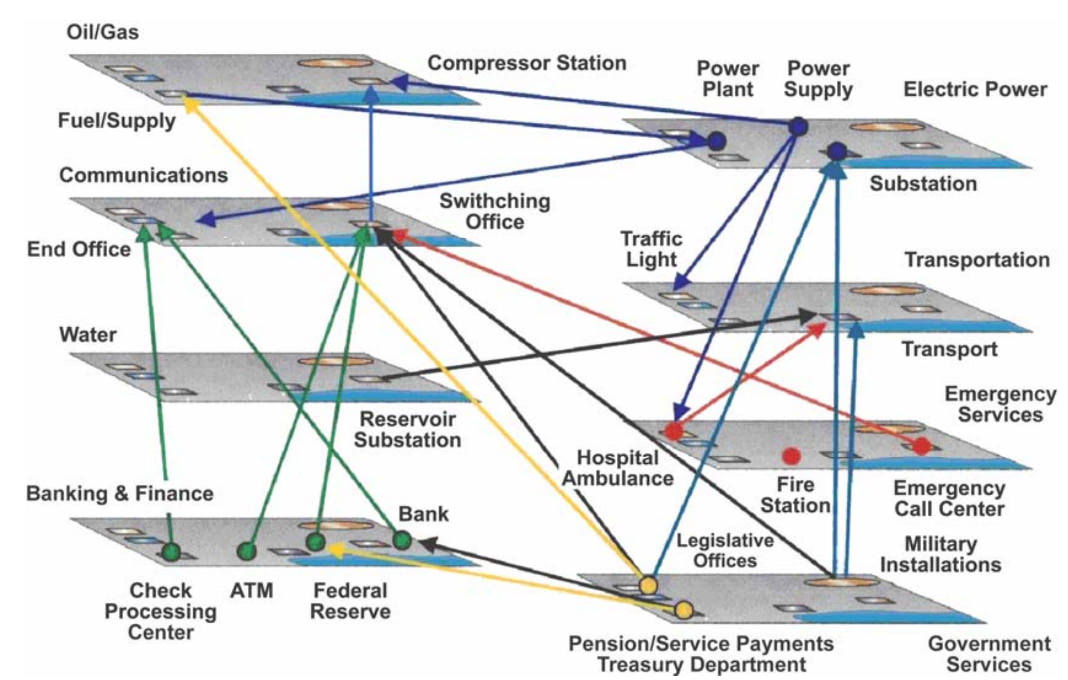
- Power-grid networks
- Communication networks
- railway networks
- Water supply
- Gas supply
- Transportation and fuel



Peerenboom, Fisher, and Whitfield, 2001

Motivation

- To understand correlated failure
- To assess risk of interdependency
- To design robust interdependent networks against attack and random failure

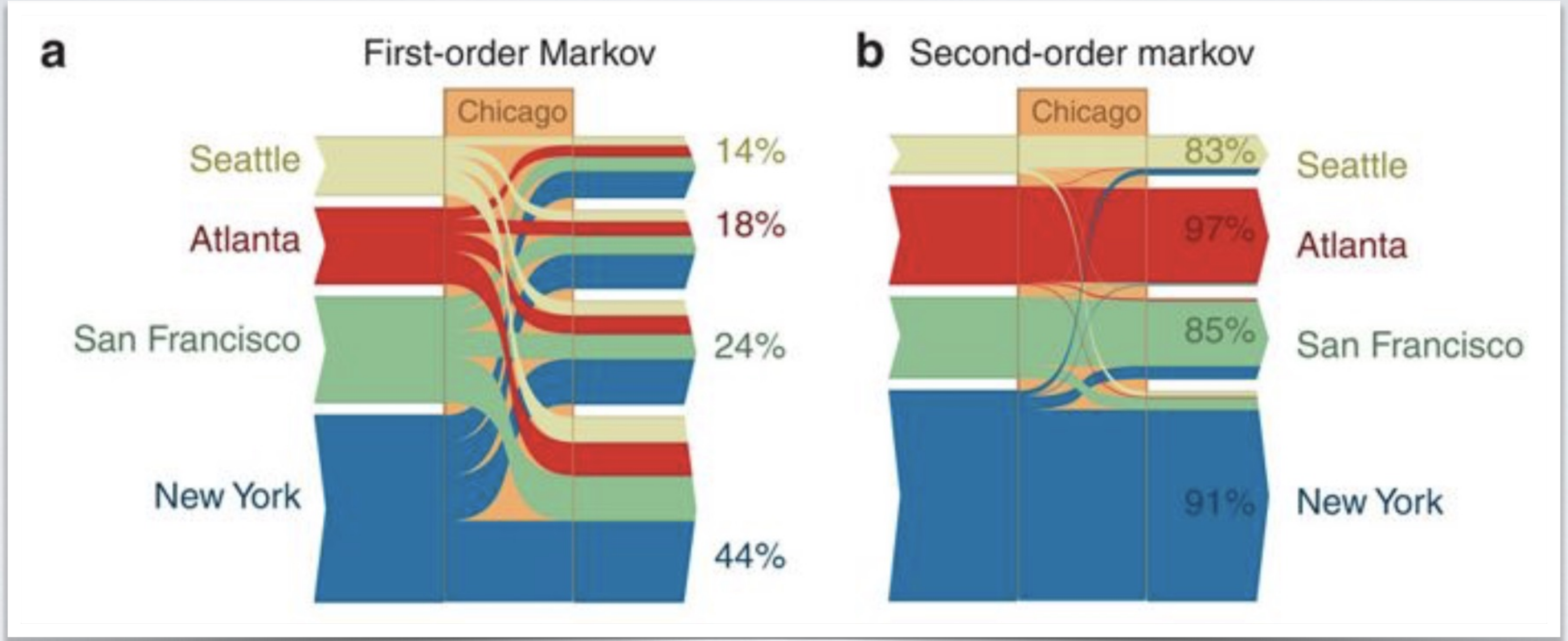


HIGHER ORDER NETWORKS (HON)

HIGHER ORDER NETWORKS

- Many networks are built using logs of *sequence of items* encountered by *actors*
    - People travelling in public transport go through stations
    - Consumer buy products on amazon one after the other
- Normal network: split sequences in pairs
    - Higher order: conserve the memory of previous items
    - From first-order Markovian to second-order Markovian

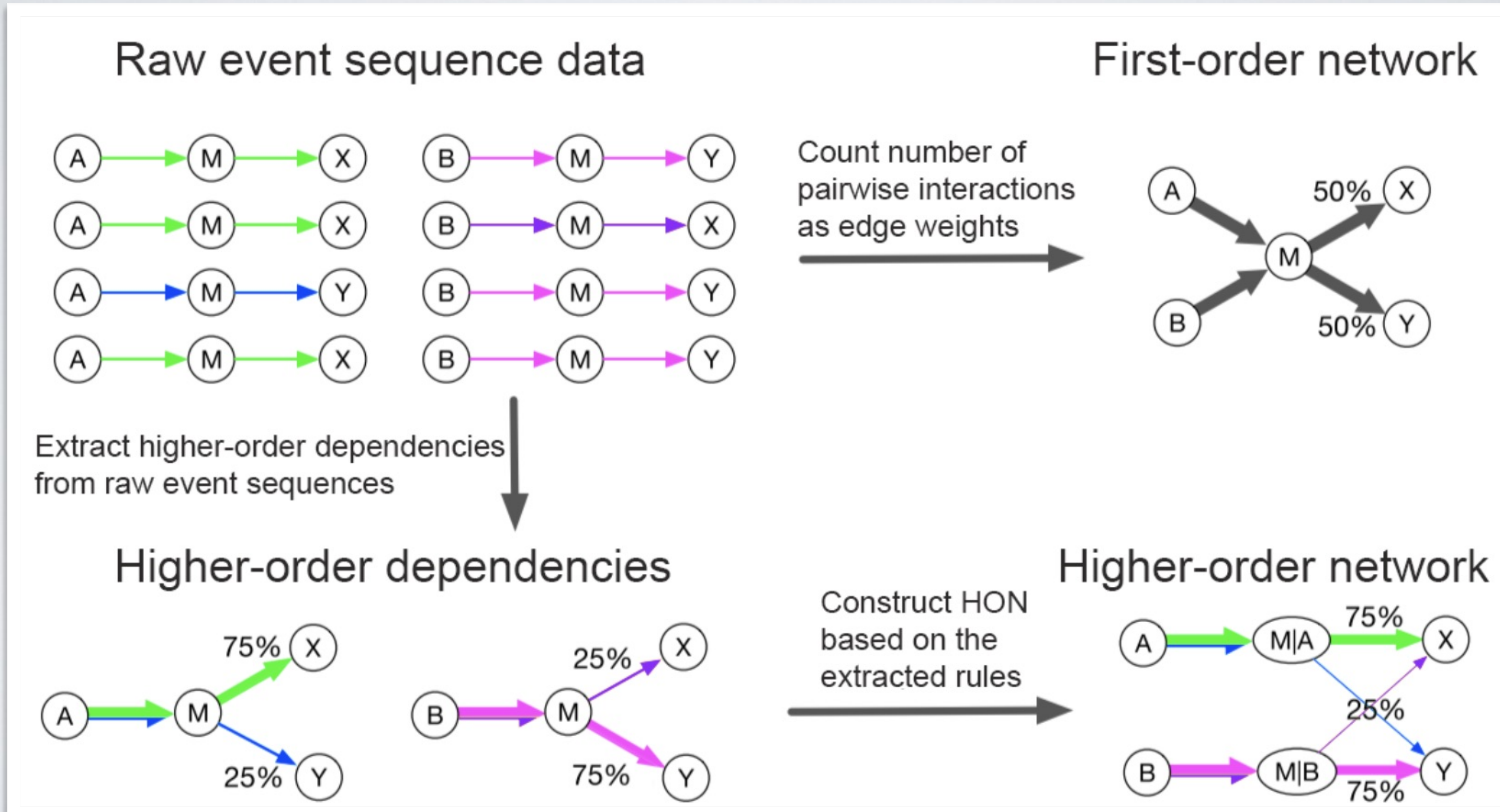
HIGHER ORDER NETWORKS



Round trips

Rosvall, M., Esquivel, A. V., Lancichinetti, A., West, J. D., & Lambiotte, R. (2014). Memory in network flows and its effects on spreading dynamics and community detection. *Nature communications*, *5*, 4630.

HIGHER ORDER NETWORKS



HIGHER ORDER NETWORKS

- Random walk approaches generalize naturally to higher order networks
    - Centrality: PageRank
    - Communities: Infomap
- At each step, the *random walker* decides to follow an out-going link
- This probability can depend on the walker origin
    - Actually representing the HON or not.

HIGHER ORDER NETWORKS

- Applying a community detection algorithm to a HON
- A node is now a tuple (node, history)
- Results of a community detection algorithm:
  - Communities are composed of (node, history) vertices
  - We can go back to a traditional community partition:
    - We forget the memory part of nodes
    - Several instances of same nodes in same community
    - Same node in different communities
    - =>Overlapping communities

HIGHER ORDER NETWORKS

- Weakness: complexity
- Number of nodes multiplied by number of possible arrival sources
- => Rare cases could be ignored, current research topic

Lambiotte, R., Rosvall, M., & Scholtes, I. (2019). From networks to optimal higher-order models of complex systems. *Nature physics*, *15*(4), 313-320.