

Network Science Cheatsheet



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Graph Convolutional Networks

Disclaimer

Graph Convolutional Networks and Graph Neural Networks in general are a recent field of research, with hundreds of publications in the last few years, and scores of new papers published in every machine learning and network science conference. This class is thus only an introduction to the mechanism underlying the most simple of these approaches.

Graph Convolutional Networks

Graph Convolutional Networks^a (GCN) are Deep Neural Networks (NN) designed to work with graphs. They are based on the principle of *propagation* on networks and are especially useful for machine learning tasks such as node classification, although they can be adapted for other tasks. GCN are a subset of a more general field of research called Graph Neural Networks (GNN)^b

^aS. Zhang et al. 2019.

^bWu et al. 2020.

Artificial Neural Networks

Artificial Neural Networks are a family of machine learning algorithms loosely inspired by biological neural network. They are composed of basic units called **neurons** or **perceptrons**, that individually play a role equivalent to a logistic classifier: they can be trained through examples to combine *input features* in order to predict a value or a class. A neural network consists in stacking such perceptrons in layers: the **output** of a perceptron is the **input** of another one. Layers are separated by non-linear functions called **activation functions**. All weights are trained simultaneously, allowing to learn **non-linear**, complex functions.

Neural Network Training

Training a neural network is the same principle as training a linear/logistic regression: the goal is to find weights that minimize a loss function, i.e., a distance, for all pairs input features/target value in a training set, between the target value and the output of the NN when provided the feature values.

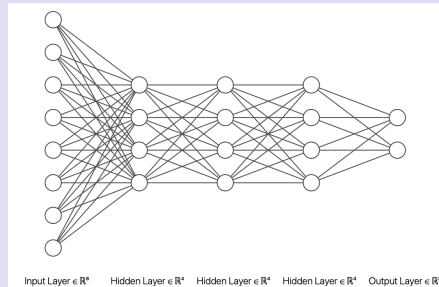
Deep Neural Networks

Deep Neural Networks is a name for neural networks composed of a large number of layers of neurons: for instance, a 4-10-2 neural network is composed of 3 layers, the first one composed of 4 neurons (receiving as input data features), whose outputs are the input of 10 neurons in the second layer, whose output are the inputs of 2 neurons in the third and last layer.

Fully Connected Layer

Several types of layers exist. The simplest one is called a Fully connected layers, and is simply defined such as all the outputs of each neuron in a layer are inputs of the neurons of the next layer.

Illustration of a Neural Network

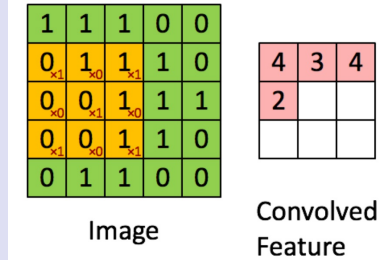


A deep neural networks composed of 3 hidden layers. The input is provided as 8 features, each of the hidden layer has 4 neurons, and they are fully connected layers. The output consists of 2 output values.

Convolution Layer

Convolution layers are another type of layers in which inputs are combined in a particular way, based on some relation known *a priori* between them. They are typically used on images: each pixel of the image is a feature (e.g.: from 0 -white- to 255 -black), and those pixels have a particular organization: each pixel has 4 (rook adjacency) or 8 (Queen adjacency) neighbors. In a convolution layer, a **higher level** pixel is created for each input pixel by combining its neighbors in a particular way. The matrix defining how to combine neighbors is called a **kernel**.

Illustration of a convolution



Example of kernels

Edge detection	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	
	$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	
Sharpen	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	
	$\frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	

^a[https://en.wikipedia.org/wiki/Kernel_\(image_processing\)](https://en.wikipedia.org/wiki/Kernel_(image_processing))

Trainable kernels

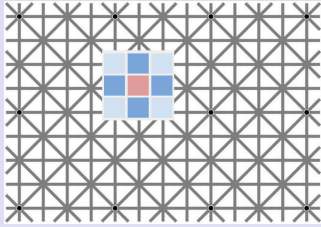
In neural networks, the kernel used in a layer is not chosen *a priori*, its weights are learned to minimize the global prediction error, as any other weight.

Convolution - intuition

The intuition of stacking several convolution layers is to learn higher levels of abstraction at every layer. Intuitively, the first layer might recognize lines and curves, the second circles and simple patterns, the third, eyes, letters or composed shapes, until the last one can recognize animals or faces, for instance.

Images as graphs

Intuitively, the relation between pixels in an image can be represented by a graph: each pixel is represented by a node, and each node is connected to neighboring pixels in the picture. The corresponding graph is thus a regular grid.



The graph representing pixels of an image. From ^a

^a <https://www.inference.vc/how-powerful-are-graph-convolutions-review-of-kipf-welling-2016-2/>

Graph Convolution

The concept of convolution can be extended to any graph. A **Graph convolutional layer** computes a value for each node by combining the values of its neighbors. Note that each node can have a different number of neighbors, thus instead of combining directly the values of the neighbors, the kernel combines the average values of the neighbors. e.g. if each node has 10 features, we first compute the average value among neighbors for each feature, and then the kernel controls how to combine those average values.

Convolution and diffusion

There is a strong relation between a graph convolutional layer and diffusion in networks. Training a graph convolutional layer consists in learning the best way for nodes to combine the features of their neighbors to make predictions, according to the training examples.

Stacking GCN layers

A single GCN layer combines, for each node, values of its direct neighbors. With 2 layers, the value of a node is computed from nodes at distances up to 2 in the graph. The more successive layers, the farther in the graph a node can influence other nodes. Remember that in most real graphs, the average shortest path is small, thus 5 or 6 layers can be enough for most nodes to influence most other nodes.

Graph Convolutional Layer: Definition

The typical Graph Convolutional Layer^a is defined as follows:

$$f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$$

We will decompose this equation to understand what it does.

^aKipf and Welling 2016a.

Feature Matrix H

Graph Convolution uses node *features* to make predictions on nodes. Nodes features are represented by a Matrix H . H_i corresponds to the vector of features of node i . Features can be attributes (age, political opinion, etc.) or structural properties (centralities, etc.). Matrix H is of shape $n \times d$, with n the number of nodes and d the number of features.

Feature Combination AH

The GCN layer consists for each node in combining its neighbors features. AH is the simplest way of doing this combination, it computes for each node a vector, corresponding to the sum of the vectors of its neighbors. For instance, if each node originally has 2 features, age and degree, after the AH operation, the vector H'_i of node i is composed of 2 values, the sum of age of its neighbors, and the sum of degrees of its neighbors. The matrix H' is thus of shape $n \times d$.

To avoid nodes with high degrees having systematically higher values, we usually use instead a **normalized adjacency matrix** to compute **average values**.

Adding self loops \hat{A}

Until now, we have said that a GCN combines the features of its neighbors. However, in most cases, the features of the node itself are also relevant. Let's assume that we want to predict an individual i *revenue* from available *age* and *betweenness* information. Using the neighbors age and betweenness is meaningful, but using age and betweenness of i itself is probably relevant too. We represent this by saying that each node is also a neighbor of itself, i.e., in graph terms, we add a self-loop to each node. In matrix term, we replace A by $\hat{A} = A + I$ with I the identity matrix (values on the diagonal are all 1, and all other values are 0).

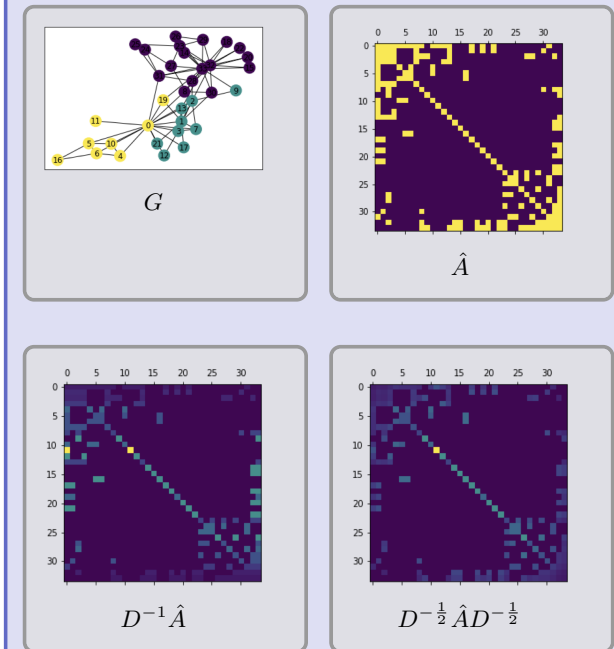
Normalized Adjacency Matrix $\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}}$

In order to compute the **average** of neighbors features instead of the sum, we can use a **normalized adjacency matrix**, by multiplying it by the inverse of the Degree matrix D^{-1} , i.e., a matrix such as $D_{ii} = \frac{1}{k_i}$, and all other values are 0.

$D^{-1} \hat{A} H$ computes the average of neighbors features. However, in many cases, it is more meaningful to use an average weighted by the degree, i.e., neighbors with small degrees have more influence than neighbors of large degrees. As for heuristics like Adamic Adar, it can be interpreted by doing a parallel with social networks: To infer a node revenue from age of neighbors, the age of my close friends (low degrees) is probably more meaningful than the age of the pop-stars (high degrees) I follow. This weighted average is computed by the normalized Adjacency matrix:

$$D^{-\frac{1}{2}} \hat{A} D^{-\frac{1}{2}}$$

$\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}}$ - visually



Weight Matrix W - Single layer

Let's note H' the result of the averaging of the neighbors features. H'_i is the vector of averaged values of node i . $H'W$ computes new features for each node by combining its neighbors features. For instance, consider that we have $d = 3$ features, e.g., age, betweenness and degree. If we have a single GCN layer and we want to predict a single variable, e.g., the revenue, then W could be of shape $d \times 1$, and each node i predicted value is $H'_{1i}w_1 + H'_{2i}w_2 + H'_{3i}w_3$, with $H'_{1i}, H'_{2i}, H'_{3i}$ respectively the average age, betweenness and degree of the neighbors of i , and w_1, w_2, w_3 learned weights, shared for every node.

Weight Matrix W sizes - Multiple layers

In most cases, GCN layers are chained one after the other. At each level, a different weight matrix is used, that we can note W_l for level l .

The size of those matrices is determined as follows: The first dimension is the number of features in **input** of the layer. The second dimension is the number of features wanted in **output** of the layer. For instance, if we originally have 10 features for each node, we can design a 3 layer neural network with weight matrix sizes: $W_1:10 \times 10, W_2:10 \times 5, W_3:5 \times 1$.

This network first transforms the 10 original features in 10 new features that are combinations of the neighbors features, then these 10 features are combined into 5, and finally those 5 features are combined into a single one.

Activation function σ

The last element of the GCN layer is the activation function σ . Activation function are commonly used between layers in deep neural networks, to add non-linearity. With GCN, the most commonly used is the **ReLU** (*Rectified Linear Unit*), which is defined as:

$$\sigma(x) = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Intuitively, it mimics the behavior of biological neurons: if the value is above a threshold, the network *fires*, and its value is propagated to the the next layer, if not, the computed value is lost.

No feature

If no feature is available for nodes, it has been proposed to use $H = I$. Since by definition $AI = A$, the average of neighbor's features of node i is A_i .

Weight initialization

The objective of the machine learning process is to learn the weight matrix W . W is typically initialized with random values, generated by a Normal distribution centered on 0. Negative values are required to benefit from the ReLU activation function.

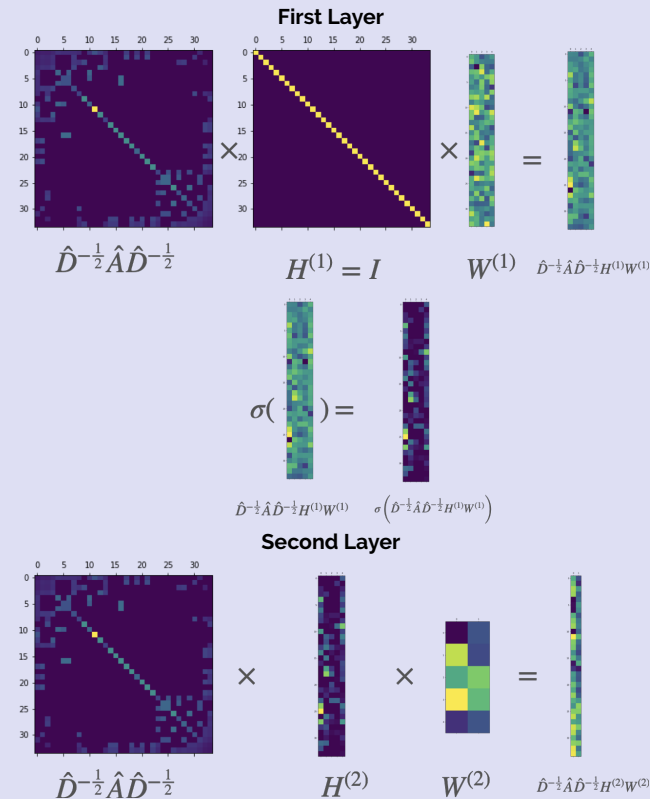
Forward and Backward steps

We call **Forward step** the computation of layers in sequence, for fixed values of W_l .

We call Backward step the update of W_l matrices. These updates are done using a technique called **back-propagation**, whose objective is to minimize the loss function (error) between the result of the forward step and the objective result (known from a training set)

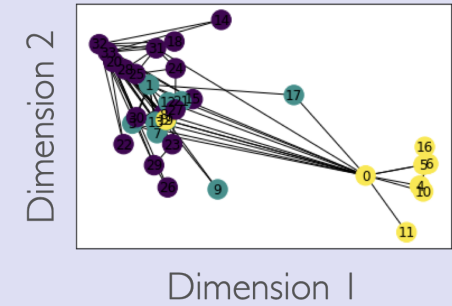
Example of a forward step

Let's look at the forward step of a simple network. The graph structure is the Zackary Karate Club. We consider no features, and thus use $H = I$. We set 2 GCN layers: l_1 converts from n dimensions to 5 dimensions, l_2 converts from 5 dimensions to 2 dimensions.



Forward step embedding

After the forward propagation, **without** weight learning, we obtain in our example a result that can be interpreted as a 2 dimensional embedding.



We can note, despite random weights, that the forward process seems to capture some network structure.

Weight learning - Backpropagation

To learn the weight matrices, one requires:

- Examples of desired outputs for some nodes (ground truth label)
- A **loss function**, i.e., a function to measure the difference between the true labels and the labels produced by the forward step

At each step, a **gradient descent** approach is used: knowing the **derivative** of the loss function, weights are updated in the direction that reduces the loss, such as a new forward step with those updated weights would produce a result closer to the true labels.

Example: ZKC node clustering

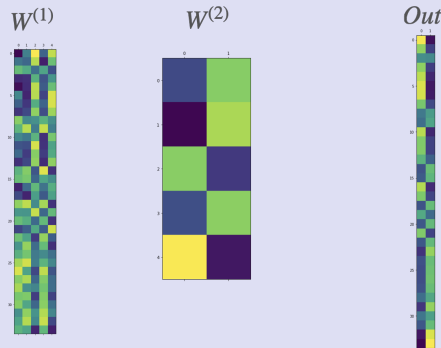
The **Zackary Karate Club (ZKC)** is a famous network dataset. It represents social relations in a Club of Karate, that were collected before the two instructors decide to split apart, each creating his own club, attracting approximately half the students of the original club. It is often used as a toy application case for community detection/Network clustering, were the task is to identify the students that will follow each teacher, from the social graph observed before the split.

ZKC as a semi-supervised Problem

To formalize the ZKC task as a classification problem, we label each of the two instructors (nodes 0 and 33) with opposed labels (0 and 1). Starting with the same setting as earlier, i.e., I matrix as features, we optimize the weights of the 2 GCN layers in order to minimize the difference between the labels of the teachers and the output of the Graph Neural Network. Note that we kept a vector of size 2 in output, with opposite objectives (for the first, instructors labels are (0,1), for the second, (1,0)).

Trained weights

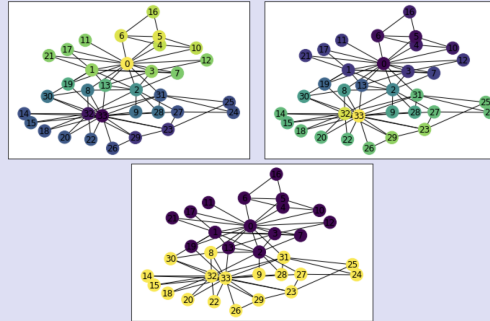
Trained weights of first and second layers, and output vectors (2 dimensions).



Note that for the first and last nodes (instructors), vectors in output correspond to the objective (respectively, (1,0) and (0,1)).

Clustering

Illustration of usages of the vectors obtained after training of the GCN.



Top Left: Values of the first computed feature. **Top Right:** Values of the second computed feature. **Bottom:** clustering obtained by assigning each node to the feature for which it has the highest value (example of heuristic).

Going Further

Python library: DGL (<https://www.dgl.ai>)
Survey GCN: (S. Zhang et al. 2019)(S. Zhang et al. 2018)
Survey Graph Neural Networks (GNN): (Zhou et al. 2018)(Wu et al. 2020)
General presentation of GNN: (Xu et al. 2018)
Variational graph Auto-Encoders (Kipf and Welling 2016b)
Link Prediction with GNN (M. Zhang and Chen 2018)
DCRNN: Convolutional Recurrent GNN (Li et al. 2017)

References

- [1] Thomas N Kipf and Max Welling. "Semi-supervised classification with graph convolutional networks". In: *arXiv preprint arXiv:1609.02907* (2016).
- [2] Thomas N Kipf and Max Welling. "Variational graph auto-encoders". In: *arXiv preprint arXiv:1611.07308* (2016).
- [3] Yaguang Li et al. "Diffusion convolutional recurrent neural network: Data-driven traffic forecasting". In: *arXiv preprint arXiv:1707.01926* (2017).
- [4] Zonghan Wu et al. "A comprehensive survey on graph neural networks". In: *IEEE Transactions on Neural Networks and Learning Systems* (2020).
- [5] Keyulu Xu et al. "How powerful are graph neural networks?" In: *arXiv preprint arXiv:1810.00826* (2018).
- [6] Muhan Zhang and Yixin Chen. "Link prediction based on graph neural networks". In: *Advances in Neural Information Processing Systems* 31 (2018), pp. 5165–5175.
- [7] Si Zhang et al. "Graph convolutional networks: a comprehensive review". In: *Computational Social Networks* 6.1 (2019), p. 11.
- [8] Si Zhang et al. "Graph convolutional networks: Algorithms, applications and open challenges". In: *International Conference on Computational Social Networks*. Springer, 2018, pp. 79–91.
- [9] Jie Zhou et al. "Graph neural networks: A review of methods and applications". In: *arXiv preprint arXiv:1812.08434* (2018).