Spatial Networks

**Definition**
A spatial network is a network in which nodes are associated to positions. The probability of observing edges between a pair of nodes depends on their distance. In most cases, the probability of being connected tends to decrease with distance, but this is not a necessary requirement.

**Position of nodes - Dimension**
The position of each node is described by a vector, i.e., a list of values. The number of values in the vector is the dimension of the space in which nodes are located. The most common space is geographical space: nodes are located by a pair (latitude, longitude). It is therefore considered a 2D space (even though earth is a sphere). But spatial networks can exist in spaces with more or less dimensions, as long as the distance between node positions is meaningful.

**Examples of 1D spaces**
- The Watts-Strogatz random graph is defined on a (circular) 1D space: each node is initially connected to its k closest nodes in this space.
- In social networks, users tend to be more connected with other users with similar age. We can consider age as a position on a 1D space. The same is true about political opinions, if we consider a Left-Right spectrum.

**Examples of 3+D spaces**
- If we consider altitude, geographical networks are 3D spaces. Neurons in the brain, atoms in proteins are also embedded in 3D spaces.
- If we consider multiple nodes properties as dimensions, nodes can be located on high dimensional spaces, e.g., age, political opinion, revenue, geographical location, etc. Be careful however, that analyzing a spatial networks needs to define the distance between nodes, which can be tricky to define if dimensions are of different natures.
- Methods such as graph embedding assign to nodes locations in arbitrary large (e.g., 128) dimensions that summarize some of the network properties (see later class).

**Distances**
The distance between each pair of nodes can be computed in different ways, depending on the nature of dimensions nodes are embedded in. The most common ones are:
- **Euclidean distance**, or \( L^2 \) distance is the usual, straight line distance.
- **Great-Circle distance** is used to measure the distance between points located on a sphere, typically the Earth for geographical data.
- **Dot product** and **Cosine Distance** are often used in high dimensions, in particular when it makes sense to multiply the location vectors.
- **Manhattan distance**, or \( L^1 \) distance, is sometimes used as a variant of Euclidean distance for high dimensional data (it is simply defined as the sum of differences in each of the dimensions.)
- Observed distances can sometimes be used, a typical example being **average time distance** in datasets of trips or traffic: the time distance between dots might be only loosely proportional to geographical distance.

**Metric Space**
In most cases, we can consider that a spatial network is embedded in a **metric space**, a space associated with a metric with properties of indiscernibility, symmetry and triangle inequality. However, this is not always the case, in particular in directed networks, in which it can be useful to consider different distances for links \((a, b)\) and \((b, a)\) (asymmetry).

**Route factor - Accessibility**
**Route Factor**, also called the detour index, measures how efficiently the network allows to go from a node to another, it is defined as the ratio between the metric distance and the route distance:

\[
Q(u, v) = \frac{\Delta_{uv}}{\ell_{uv}}
\]

**Node Accessibility**: Average route factor from a node to all others:

\[
\langle Q(u) \rangle = \frac{1}{N-1} \sum_{v \in N(u)} Q(u, v)
\]

**Accessibility**: Average route factor for the whole network:

\[
\langle Q \rangle = \frac{1}{N(N-1)} \sum_{u \neq v} Q(u, v)
\]

**Notation**
- \( \Delta_{uv} \): Metric distance between \( u \) and \( v \) (Euclidean, Manhattan, etc.)
- \( \ell_{uv} \): Route distance between \( u \) and \( v \), i.e., sum of Metric distances between nodes on the shortest path between \( u \) and \( v \)
- \( s^u \): Distance strength, cumulative distance from a node to its neighbors. \( s^u = \sum_{v \in N(u)} \Delta_{uv} \). The relation between \( \ell_{uv} \) and \( s^u \) can be studied, for instance to see if larger nodes tend to connect at longer distances.
Random Geometric Graphs (RGG)

Random Geometric graphs (RGG), also called Disk-percolation random graphs, are defined as such:

- Distribute \( n \) nodes randomly on a bounded \( d \) dimensional space.
- Connect any two nodes at distance less than a parameter \( r \).

Properties:
- **Degree distribution**: Poissonian, as ER random graphs.
- **Clustering coefficient** (in large graphs):

\[
C = \frac{3 \sum_{d=1}^{d-1} \binom{d}{2} \binom{n}{d}}{\sum_{d=1}^{d} \binom{n}{d} \binom{n-d}{d}}
\]

It does not depend on the number of nodes, unlike random graphs, thus is not vanishing with network size for fixed average degree.

\(^\circ\)Dall and Christensen 2002

Soft RGG (Waxman random graph)

Soft RGG, or Waxman Random Graph\(^\circ\), starts as the RGG by distributing nodes at random in a space, but instead of adding links between all nodes closer than a certain distance, it assigns edges between nodes according to a deterrence function \( f \), i.e., a function defining how distance affects the probability of observing edges between nodes. The Soft RGG can model an ER random graph if \( f \) is a constant function, \( f(\Delta) = p \). It can model a classic RGG if \( f \) is a threshold function with:

\[
f(d) = \begin{cases} 
1 & \Delta \leq r \\
0 & \Delta > r 
\end{cases}
\]

\(^\circ\)Waxman 1988

Ad-Hoc deterrence function

When a spatial model is used to create a randomized version of an observed network, the most appropriate deterrence function can be learned from data. A simple way to achieve this is to count the fraction of edges occurring between nodes at a given distance, and to compare it with edges that should appear at random if there was no spatial effect. To avoid overfitting (each pair of node being at different distances with infinite precision), we usually create bins of relevant size, e.g., every cm, km, 100km, etc., or using bins of exponentially growing size, e.g., \([0,1],[1,3],[3,7],[7,15],[15,31]\). More formally, the deterrence function is defined as:

\[
f(\Delta) = \frac{\sum_{i,j} A_{ij} \delta_{\{d_{ij}\leq \Delta\}}}{\sum_{i,j} M_{ij}}
\]

with \( A_{ij} \), the adjacency matrix (or weight matrix) of the observed graph and \( M_{ij} \), the probability of observing an edge (or weight of edges) between nodes \( i \) and \( j \) according to the chosen null model. For instance, with the simplest hypothesis that edges occur completely at random, \( \forall i,j, M_{ij} = d \) (with \( d \) the network density).\(^\circ\)Dall and Christensen 2002

Deterrence function

A deterrence function defines how the distance affects the probability of observing an edge. It can be a probability (bounded on \([0,1]\)), or define a change ratio.

1. It can be defined \( \alpha \) priori, usually as a classic monotonically decreasing function, e.g., Negative exponential \( f(\Delta) = e^{-\alpha \Delta} \) or Negative power \( f(\Delta) = \Delta^{-\alpha} \), with \( \alpha \) a parameter. A typical example of negative power in geographical data is when the probability of observing an edge decreases as the square of the distance, i.e., \( f(\Delta) = \frac{1}{\Delta^2} \).

2. It can also be learned from data, either by fitting parameters of a predefined function (e.g., the \( \alpha \) parameter above), or by using an Ad-Hoc deterrence function.

Gravity Model of Spatial Interactions

The Gravity Model of Spatial Interaction has been known for a long time in Geography. It is defined by analogy with Newton’s law of gravitation and, in its original form, says that the strength of the interaction between two places (countries, cities, etc.) is proportional to their power of attraction \( P_i \) and to the inverse of their distance. More formally, the expected strength of interaction \( G_{ij} \) between locations \( i \) and \( j \) is:

\[
G_{ij} = K \frac{P_{i}^{out} P_{j}^{in} \Delta_{ij}}{\Delta_{ij}^3}
\]

Common examples would be a model of a job market between cities, with \( P_{i}^{out} \), the number of jobs offered in city \( i \) and \( P_{j}^{in} \) the number of job seekers in city \( i \). \( K \) is a normalization constant.

\(^\circ\)Cazabet, Borgnat, and Jensen 2017

Degree-Preserving Gravity Model

A weakness of the network gravity model is that it does not preserve degrees: if we consider two nodes for which we have observed a same degree \( k \), one located in the center of space, and thus having many other nodes at positively biased distances, and the other at the periphery having fewer nodes at those distances, the peripheral node will having fewer edges according to the network model than the central one. A solution to correct this is to fit nodes attractiveness that would best explain the observed degrees, for a given graph and a given deterrence function.
The Radiation Law of Spatial Interactions is another random spatial model. Unlike previous ones, it does not depend on a deterrence function, and is parameter-free. It is based on the principle of relative opportunities: the probability of observing an interaction from \(i\) to \(j\) depends on \(P_{out}^i\), \(P_{in}^j\), and the sum of all \(P_{in}^k\) for \(\Delta_{ik} < \Delta_{ij}\), i.e., other opportunities accessible at a shorter distance. More formally:

\[
R_{ij} = k_i^{out} \frac{P_{out}^i P_{in}^j}{(P_{out}^i + s_{ij})(P_{out}^i + P_{in}^j + s_{ij})}
\]

With \(s_{ij} = \sum_{u \in V, \Delta_{iu} < \Delta_{ij}} P_{in}^u\) the sum of opportunities at a shorter distance than the target.

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Illustration of the zone \(s_{ij}\) in which opportunities decrease the probability of interactions between \(i\) and \(j\). Map of Belgium. Black Line: communities without spatial correction/geographic partitions. Colors: Space-corrected partition (linguistic partition).

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Simini et al. 2012

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The Radiation Law VS Gravity Law

The advantage of the radiation law compared with the gravity law is that two nodes located at the same distance and of similar degrees can have different edge probabilities depending on their surroundings. Intuitively, the expected relation between two small scale cities at distance \(l\) is different if both cities are far from any other large town, or if a Metropolis lies between them.

On the contrary, the weakness of the Radiation Law comes from its simplicity: without deterrence function, it is impossible to take into account non-linear and non-monotonic influence of the distance.

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Space-Corrected Community Detection

Community detection applied to spatial networks tends to yield communities corresponding to a spatial partition of space, even if there is actually no boundary between those regions. A method as been proposed to remove the influence of space, and thus discover communities corresponding to non-spatial (social, etc.) effects, usually hidden behind the influence of spatial constraints. The principle is to use a Modularity-maximization algorithm, in which the null-model used by Modularity (usually, a Configuration Model) is replaced by a spatial model (usually, a Gravity Model).

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Expert et al. 2011

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References