M2 Complex Systems - Complex Networks

Lecture 3 - Network models

Erdös-Rényi random graphs and configuration model

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* Thanks to Daron Acemoglu and Asu Ozdaglar for pedagogical material used for these slides.

Network models

Model = **random generation** of synthetic networks

- To simulate :
  - phenomena
  - algorithms
  - protocols
- In order to :
  - design
  - test
  - predict
  - better understand
- Example :

  Would Internet protocols still work if Internet was 10 times larger ?
  - generate a synthetic network and simulate

Network models

Model = **random generation** of synthetic networks

... having the properties of real-world networks ! ! !

Network models

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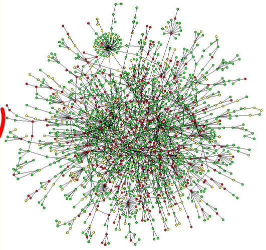
... having the properties of real-world networks ! ! !

Four classic properties of real-world complex networks :

- Low global density

$m = \#\text{edges} \ll \frac{n(n-1)}{2}$

$\frac{m}{\frac{n(n-1)}{2}} \ll 1$



10^{-6} 10^{-5}

Network models

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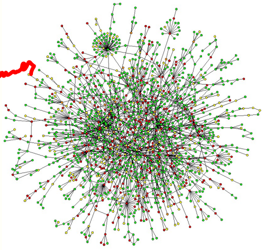
Four classic properties of real-world complex networks :

- Low global density
- Short distances

↦ randomness

average distance ~ cte

(diameter) ~ log n

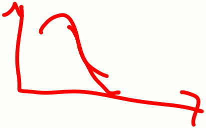


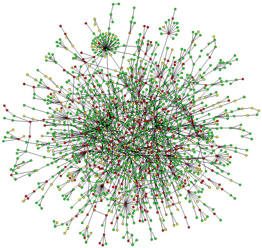
Network models

Model = **random generation** of synthetic networks
... having the properties of real-world networks ! ! !

Four classic properties of real-world complex networks :

- Low global density
- Short distances
- Heterogeneous degrees





Network models

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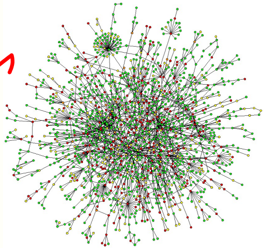
Four classic properties of real-world complex networks :

- Low global density
- Short distances
- Heterogeneous degrees
- High local density



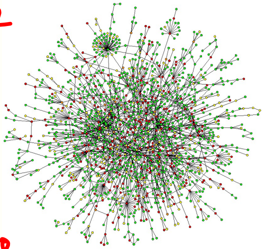
Network models

Model = **random generation** of synthetic networks
… having the properties of real-world networks ! ! !

Four classic properties of real-world complex networks :

- Low global density   0,000342
- Short distances   3,45
- Heterogeneous degrees
- High local density

Clust coef = 0,08



Goal : generate synthetic networks having these four properties (in a generic way)

Erdös-Rényi (ER) random graphs

There are two models :

- $G_{n,m}$: choose uniformly at random (u.a.r.) m edges among the n vertices

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$$p \frac{n(n-1)}{2} = m$$

$$p = \frac{2m}{n(n-1)}$$

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Erdös-Rényi (ER) random graphs

1959 $G_{n,m}$

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Should we use $G_{n,m}$ or $G_{n,p}$? → $O(n^2)$

- For generating networks ? $G_{n,m}$ $O(m)$
- For mathematical analysis of the model ? $G_{n,p}$

$G_{n,m}$: implementation and complexity $O(m)$

- Algo : Pick m times two vertices uniformly at random

$0 \ldots n-1$ $\quad a = \text{rand}(0, n-1)$ $\quad (a, b)$

$b = \text{rand}(0, n-1)$

$(a\ b)$ risk of collision → multiple edges

$(a\ a)$ self loop

Pick until you get m distinct edges.

$O(m)$ time.

$G_{n,m}$: implementation and complexity

- Algo : Pick m times two vertices uniformly at random
  - How to deal with self-loops ?

$G_{n,m}$: implementation and complexity

- Algo : Pick m times two vertices uniformly at random
  - How to deal with self-loops ?
  - How to deal with multiple edges ?

Properties of $G_{n,p}$

Four properties to check :

- Low global density
  - p parameter of the model, controls m : $\mathbb{E}(m) = \frac{pn(n-1)}{2}$

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Four properties to check :

- Low global density ✓
  - p parameter of the model, controls m : $\mathbb{E}(m) = \frac{pn(n-1)}{2}$
  - law of large numbers : m is very concentrated around its mean

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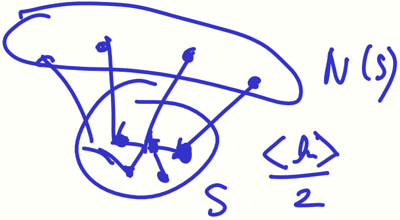
Properties of $G_{n,p}$



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Expansion property

- def. (graph theory) : a graph G is a c-vertex-expander iff $\forall S \subseteq V$ s.t. $|S| \leq \frac{|V(G)|}{2}$, we have $|N(S)| \geq c \cdot |S|$

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  - expansion of $G_{n,m}$? $\sim \frac{m}{n}$ $\frac{2m}{n} = \langle k \rangle$

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Properties of $G_{n,p}$

Four properties to check :

p(n)

- Low global density ✓
- Short distances ✓

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  - fix the average degree $\lambda = p(n-1)$

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  - fix the average degree $\lambda = p(n-1)$
  - $$\begin{aligned}\mathbb{P}(d^\circ = k) &= \binom{n-1}{k} p^k (1-p)^{(n-1-k)} \\ &= \frac{A_n^k}{k!} \frac{\lambda^k}{(n-1)^k} \left(1 - \frac{\lambda}{n-1}\right)^{n-1-k} \\ &= \frac{A_n^k}{(n-1)^k} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n-1}\right)^{n-1} \left(1 - \frac{\lambda}{n-1}\right)^{-k}\end{aligned}$$

k: fix
λ: fix
$n \to +\infty$

$\downarrow 1$ $\quad \downarrow e^{-\lambda}$ $\quad \downarrow 1$

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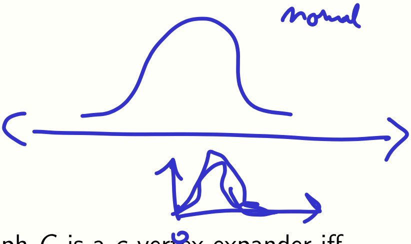

- Heterogeneous degrees ✗
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  - then when $n \to +\infty$, $\mathbb{P}(d^\circ = k) \to \frac{\lambda^k e^{-\lambda}}{k!}$: Poisson law

Properties of $G_{n,p}$

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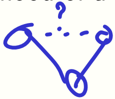

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- Heterogeneous degrees ✗
- High local density ✗
  - probability of an edge in the neighbourhood of a vertex ?
  - same as everywhere : p (couples of vertices are independant)

Phase transitions in $G_{n,p}$

N.B. : p (eventually) depends on n

- Threshold function $t(n)$ for property A :
  - $\mathbb{P}(A) \to 0$ if $\frac{p(n)}{t(n)} \to 0$
  - $\mathbb{P}(A) \to 1$ if $\frac{p(n)}{t(n)} \to +\infty$
  - makes sense for monotonic properties (for inclusion of edge set)

$p(n) = o(t(n))$

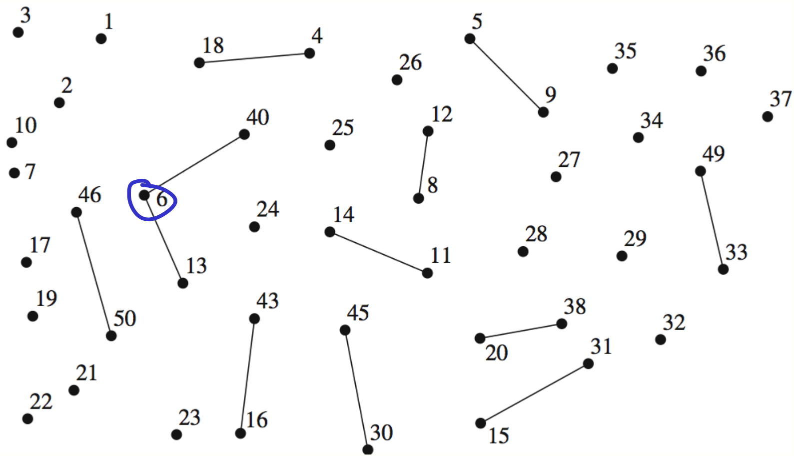
$t(n) = o(p(n))$

- such a threshold function exists $\Rightarrow$ **phase transition**

- Seminal work of Erdös and Rényi in 1959

Phase transitions in $G_{n,p}$

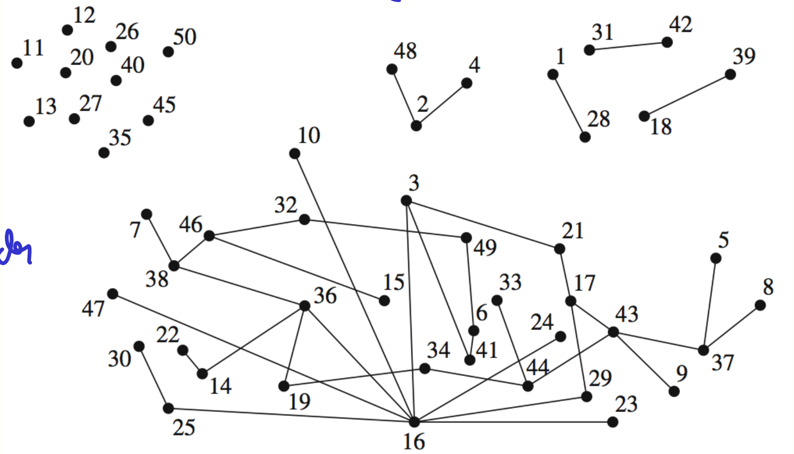
$n = 50,\ p = 0.01$



Phase transitions in $G_{n,p}$

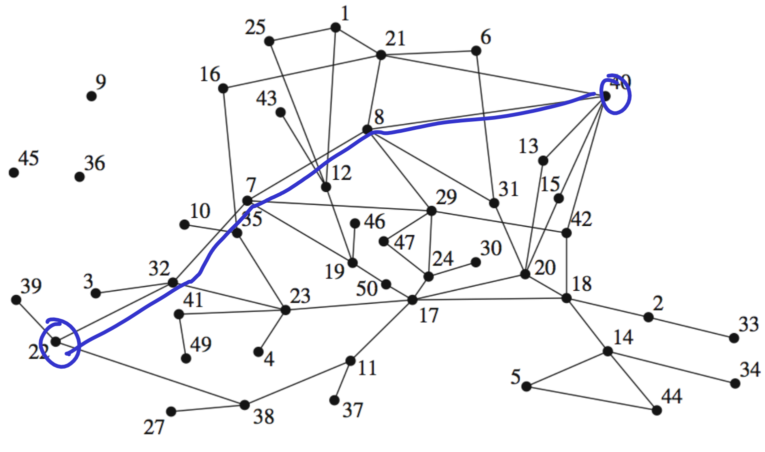
$n = 50,\ p = 0.03$

cycles



Phase transitions in $G_{n,p}$

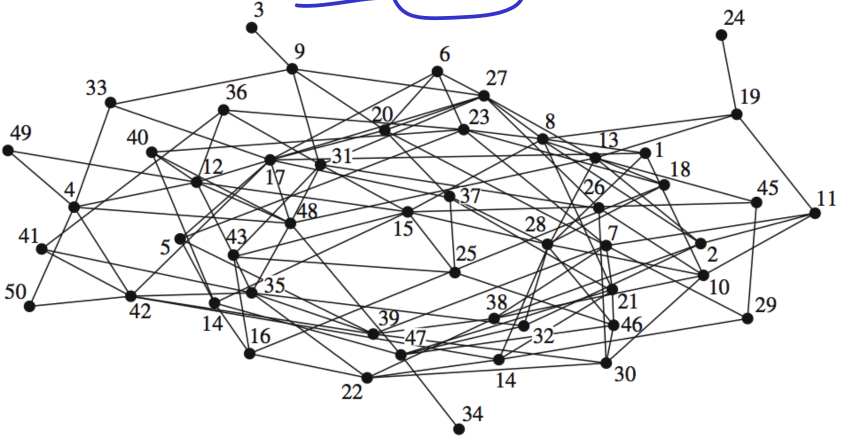
$n = 50,\ p = 0.05$



giant compont..

Phase transitions in $G_{n,p}$

$n = 50, p = 0.10$



Threshold for connectivity

- We show a threshold with function $t(n) = \frac{\log n}{n}$
- Denote $p(n) = \lambda \frac{\log n}{n}$ (mean degree $\sim \lambda \log n$)
- We show a (much) stronger statement for threshold function $\frac{\log n}{n}$:
  1. $\mathbb{P}(connectivity) \to 0$ if $\lambda < 1$
  2. $\mathbb{P}(connectivity) \to 1$ if $\lambda > 1$

Proof of (1)

$\lambda < 1$ $\quad p(n) = \lambda \frac{\log n}{n}$

- to prove (1), it is enough to show that the probability of existence of at least one isolated node goes to 1

0

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- Let I_i be the Bernouilli random variable defined as
  - $I_i = 1$ if vertex i is isolated
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- Let $X = \sum_{i=1}^{n} I_i$ be the number of isolated vertices
- We have $\mathbb{E}[X] = n.n^{-\lambda} \to +\infty$ for $\lambda < 1$

$\frac{n}{n^{\lambda}}$

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$\Rightarrow$ NO, we need a concentration property.

Proof of (1)

$X = \sum I_i$

- We have

$$\begin{aligned} var(X) &= \sum_i var(I_i) + \sum_i \sum_{j \neq i} cov(I_i, I_j) \\ &= nvar(I_1) + n(n-1)cov(I_1, I_2) \end{aligned}$$

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- And we also have :
  - $var(I_1) = \mathbb{E}[I_1^2] - \mathbb{E}[I_1]^2 = q - q^2$
  - $cov(I_1, I_2) = \mathbb{E}[I_1 I_2] - \mathbb{E}[I_1]\mathbb{E}[I_2]$
  - $\mathbb{E}[I_1 I_2] = \mathbb{P}(I_1 = 1, I_2 = 1) = (1-p)^{2n-3} = \frac{q^2}{1-p}$

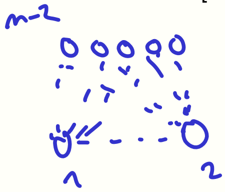
$E(X) = \frac{n}{2}$

$E(X^2) \sim n^3$

$\frac{n^2}{4}$

$q = (1-p)^{n-1}$

$i^k \quad n^{k+1}$



$$\sum_{i=1}^{n} i^2 \sim n^3$$

$$\sum_{i=1}^{n} i \sim n^2$$

$2(n-2) + 1$

Proof of (1)

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  - $\mathbb{E}[I_1 I_2] = \mathbb{P}(I_1 = 1, I_2 = 1) = (1-p)^{2n-3} = \frac{q^2}{1-p}$
- We then obtain $var(X) = nq(1-q) + n(n-1)\frac{q^2 p}{1-p}$

Proof of (1)

$p = \lambda \frac{\log n}{n}$

- We have

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$q = \left(1 - \lambda \frac{\log n}{n}\right)^{n-1} \leq e^{-\lambda \log n}$

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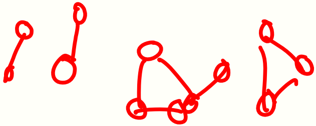
- We then obtain $var(X) = nq(1-q) + n(n-1)\frac{q^2 p}{1-p}$
- when $n \to +\infty$, then $q \to 0$ and $p \to 0$
- this gives
$$\begin{aligned} var(X) &\sim nq + n^2 q^2 p \\ &= n n^{-\lambda} + \lambda n \log n n^{-2\lambda} \\ &\sim n n^{-\lambda} = \mathbb{E}[X] \end{aligned}$$

Proof of (1)

- so we have $var(X) \sim \mathbb{E}[X]$
- and because $var(X) \geq (0 - \mathbb{E}[X])^2 \mathbb{P}(X = 0)$
- we obtain $\mathbb{P}(X = 0) \leq \frac{1}{\mathbb{E}[X]} \to 0$
- it follows that $\mathbb{P}(X > 0) \to 1$ when $n \to +\infty$
- and consequently $\mathbb{P}(disconnected) \to 1$ when $n \to +\infty$

Proof of (2)



- we now fix $\lambda > 1$
- let's check that $\mathbb{E}[X] = nn^{-\lambda} \to 0$ when $n \to +\infty$ ✓

Proof of (2)

- we now fix $\lambda > 1$
- let's check that $\mathbb{E}[X] = nn^{-\lambda} \to 0$ when $n \to +\infty$ ✓

- observe that G is disconnected $\iff \exists k$ vertices without edges to the other vertices, for some $k \leq n/2$

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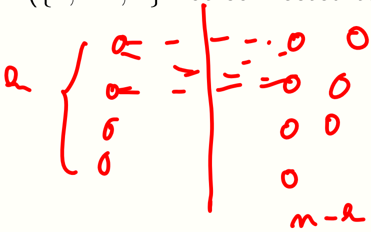

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- using this expression, one can show that $\mathbb{P}(G \text{ is disconnected}) \to 0$ when $n \to +\infty$

Threshold for giant component

- Giant = constant fraction of the vertices
- We show a threshold with function $t(n) = \frac{1}{n}$
- Denote $p(n) = \frac{\lambda}{n}$ (mean degree $\sim \lambda$)
- We again show a strong statement for threshold function $\frac{1}{n}$:
  1. if $\lambda < 1$, $\forall a \in \mathbb{R}_+^*, \mathbb{P}(maxsize(CC) \geq a \log n) \to 0$
  2. if $\lambda > 1$, $\exists b \in \mathbb{R}_+^*, \mathbb{P}(maxsize(CC) \geq b.n) \to 1$

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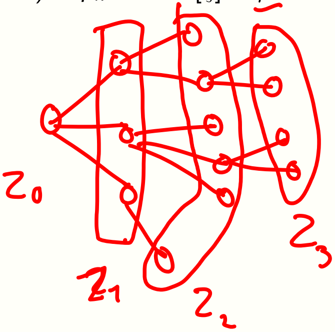

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  - and by recursion, for $k \geq 1$, we obtain
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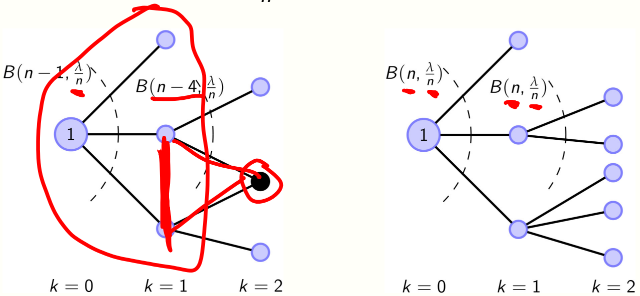

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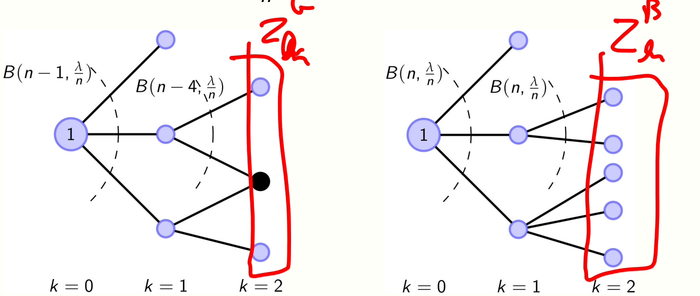

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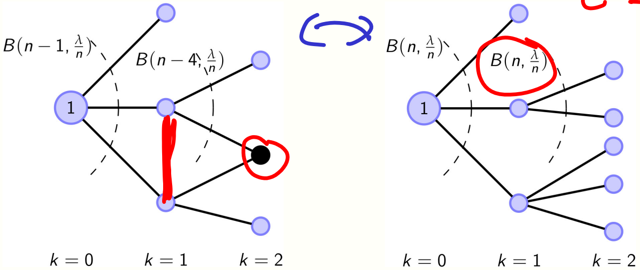

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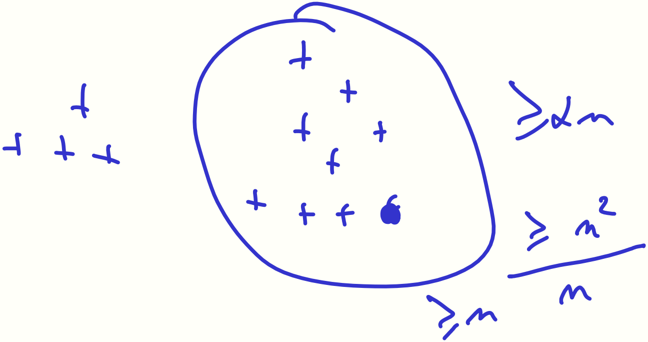

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- so if $\lambda < 1$, the expected size of the components of vertex i is constant $\Longrightarrow$ no giant component
- one can show (not shown here) that the size of the bigger component does not exceed $\log n$:

$$\forall a > 0, \mathbb{P}(max_{1 \leq i \leq n} |S_i| \geq a \log n) \to 0 \text{ as } n \to +\infty$$

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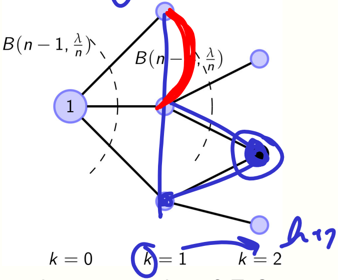

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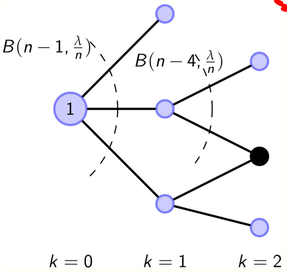

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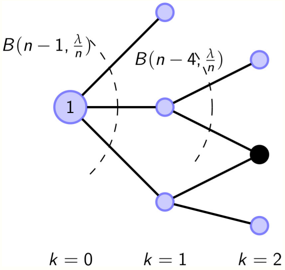

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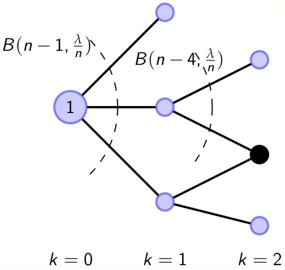

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$\Longrightarrow$ $\mathbb{E}[\#\text{conflicts}]$ becomes $\Omega(1)$ only when $\lambda^k \approx \sqrt{n}$

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- $\mathbb{E}[S_i] = \sum_k \mathbb{E}[Z_k^G] \geq \sum_{k \leq \log_\lambda(\sqrt{n})} \mathbb{E}[Z_k^B] = \sum_{k \leq \log_\lambda(\sqrt{n})} \lambda^k$
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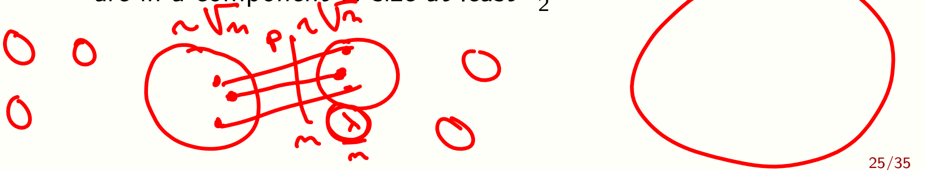
- for large n, we obtain $\mathbb{P}(|S_i| \geq \frac{\sqrt{n}}{2}) \geq cte$
  $\Longrightarrow$ there is a constant fraction of the nodes (say $\alpha.n$) that are in a component of size at least $\frac{\sqrt{n}}{2}$



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- since $|A| \geq \alpha.n$, this constitutes a giant component

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- The analysis of function Φ shows that it has a unique fixed point $\rho^* \in]0, 1[$

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- the probability for two such balls not to be connected by an edge is $(1-p)^{\frac{n}{\log^2 n}} \leq e^{-\frac{1}{\log n}}$

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  - What about edges between vertices of Z_k ?
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- Then $|S_i| \approx (\log n)^{\frac{\log n}{2 \log \log n} - 1} = \frac{\sqrt{(\log n)^{\frac{\log n}{\log \log n}}}}{\log n} = \frac{\sqrt{n}}{\log n}$
- One can cover the vertex set by approx. $\sqrt{n} \log n$ balls of size approx. $\frac{\sqrt{n}}{\log n}$
- the probability for two such balls not to be connected by an edge is $(1-p)^{\frac{n}{\log^2 n}} \leq e^{-\frac{1}{\log n}}$
- so the proba for them to be connected is at least $1 - e^{-\frac{1}{\log n}} \sim \frac{1}{\log n}$

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- Therefore, between any two vertices of G there exists with probability tending to 1 when $n \to +\infty$ a path of length $(\frac{\log n}{2\log\log n} - 1) + 1 + 2(\frac{\log n}{2\log\log n} - 1) + 1 + (\frac{\log n}{2\log\log n} - 1) \leq \frac{2\log n}{\log\log n}$

Configuration model – Molloy & Reed 1995

Input : an arbitrary degree distribution

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What degree distribution should we take as parameter ?

- The degree distribution of some real-world network
- A mathematically defined one, powerlaw $\mathbb{P}(k) \sim k^{-\alpha}$.

Configuration model : implementation and complexity

- Put the semi-links in a table of size $2m$
- Pick m times two of them uniformly at random

Properties of the configuration model

Four properties to check :

- Low global density ✓
  - the degree distribution is the parameter of the model and controls m : $m = \frac{\sum_{0 \leq k \leq n-1} k.N_k}{2}$

Properties of the configuration model

Four properties to check :

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  Expansion property :

  - Degree of the extremity of one edge :
    $\mathbb{P}(d^\circ(ext) = k') = \frac{k'\mathbb{P}(k')}{<k>}$
  - Probability that following one edge leads to k new vertices :
    $q(k) = \mathbb{P}(d^\circ(ext) = k + 1)$
  - Expected number of new vertices following one edge :
    $\sum_k kq(k) = \frac{<k^2> - <k>}{<k>}$

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- Low global density ✓
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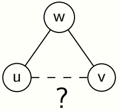

- ▶ Probability to have a link between u and k with $d^\circ(u) = k$ and $d^\circ(v) = k'$: $\mathbb{P}(uv|kk') = \frac{kk'}{<k>N}$

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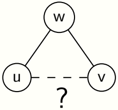

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$$\begin{aligned} \mathbb{P}(triangle) &= \sum_{k\geq 1}\sum_{k'\geq 1}\frac{kk'}{<k>N}q(k)q(k') \\ &= \frac{1}{<k>N}\sum_{k\geq 1}kq(k)\sum_{k'\geq 1}k'q(k') \\ &= \frac{1}{N}\frac{(<k^2>-<k>)^2}{<k>^3} \end{aligned}$$

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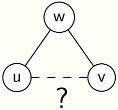

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