M2 Complex Systems - Complex Networks

Lecture 5 - Community detection algorithms

Girvan-Newman, Louvain, Leiden

Automn 2021 - ENS Lyon

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Communities in complex networks

What is a community ?

"Moral" definition

- A group of nodes that share something...
  - People with a common interest
  - Web pages with similar content
  - Proteins realising a common function

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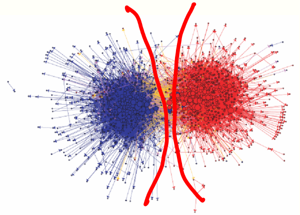
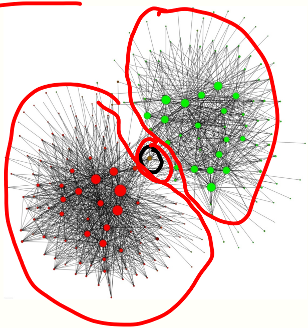
  - Proteins realising a common function
- ... that makes them be in relationship in the network!



Political blogs in US



Languages in Belgium

Communities in complex networks

What is a community?

Structural definition

- A highly connected group of nodes

Communities in complex networks

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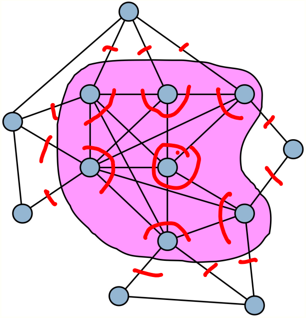
  - Density inside the community much higher than global density of the network

Communities in complex networks

What is a community ?

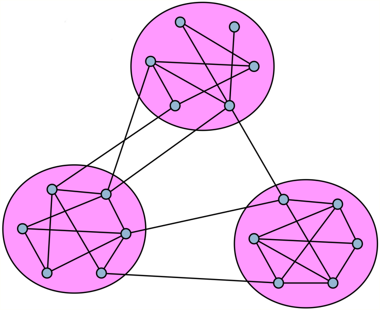
Structural definition

- A highly connected group of nodes
  - Density inside the community much higher than global density of the network
  - Only few edges toward the rest of the network



Types of structural communities

- Partition of the nodes into dense parts sparsely connected between them
  - High density inside communities
  - Few edges between communities



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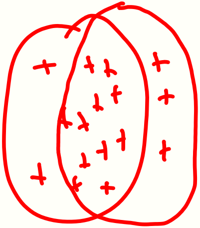
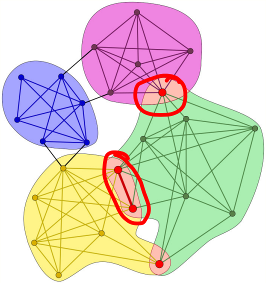
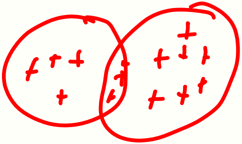
- Overlapping communities
  A node can belong to several communities
  - more realistic
  - problem : how to separate communities ?







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  - High density inside communities
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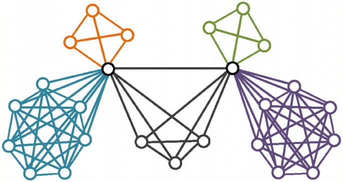
  - problem : how to separate communities ?
- Partition of the links
  - a link belong to exactly one community
  - a node can have links in different communities



Partition of the nodes

Various approaches, among them :

- random walks
- spectral methods
- hierarchical clustering
- divisive methods
- Louvain, Leiden

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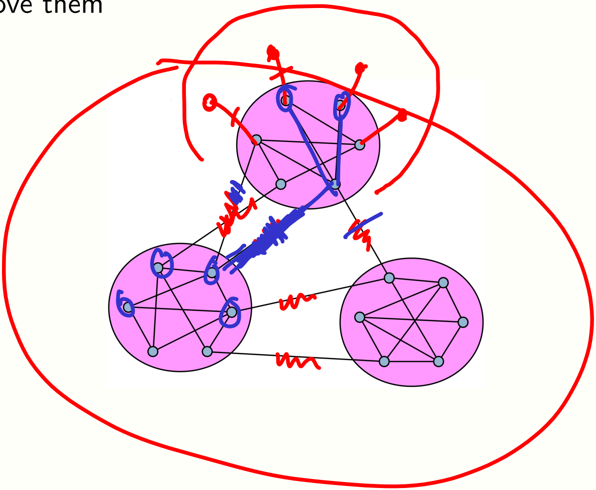
Divisive approach : Girvan & Newman 2002

The idea :

1. identify inter-community links
2. remove them



How to identify inter-community links?

- Betweenness centrality of links
  - $C_B(e) = \sum_{s \neq t} \frac{\sigma_{st}(e)}{\sigma_{st}}$ where
    - $\sigma_{st} = \#$ shortest paths from s to t
    - $\sigma_{st}(e) = \#$ shortest paths from s to t containing e

$$C_B(u) = \sum_{s \neq u \neq t} \frac{\sigma_{st}(u)}{\sigma_{st}}$$

How to identify inter-community links ?

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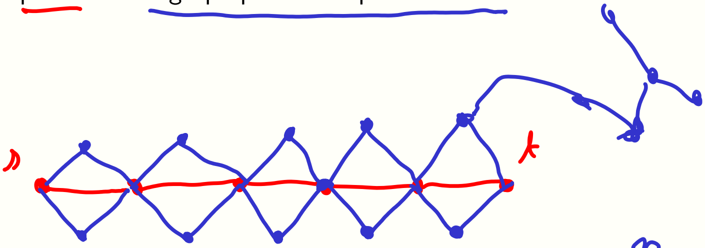
  - high betweenness $\Leftrightarrow$ e is on a high proportion of shortest paths for a high proportion of pairs of nodes



$2 \times 2 \times 2$

$2^{\frac{n}{2}}$

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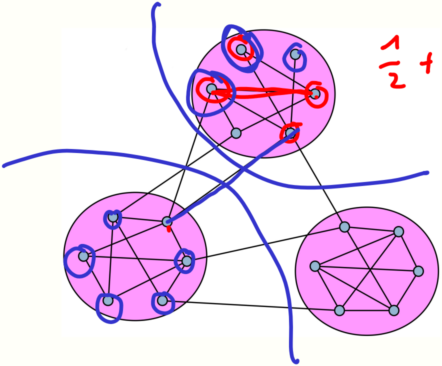

$\frac{1}{2} + 1 + 1 = 2.5$

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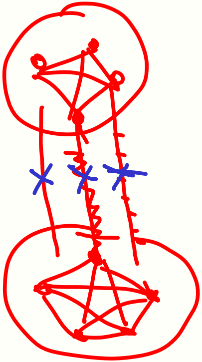

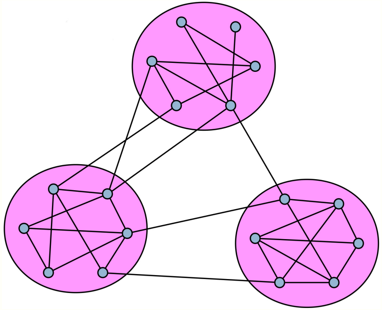

The algorithm (?)

- Algo Girvan-Newman(G)
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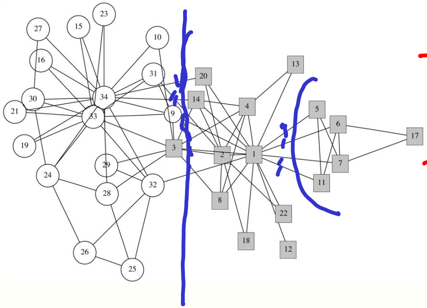

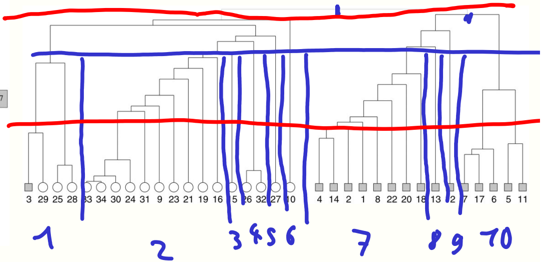

The algorithm

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  1. Compute the betweenness centrality of all links e of G
  2. for all links e in decreasing betweenness centrality do
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     - **update the betweenness centrality of all links**
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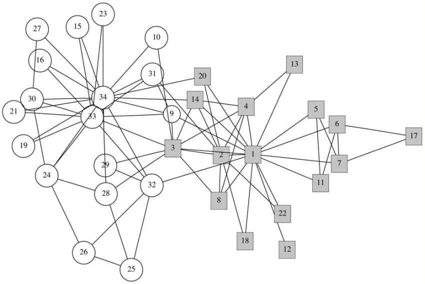
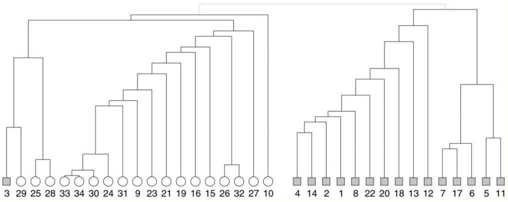




The algorithm

- Algo Girvan-Newman(G)
  1. Compute the betweenness centrality of all links e of G
  2. for all links e in decreasing betweenness centrality do m iter
     - remove e from G
     - update the connected components of G $O(m)$
     - update the betweenness centrality of all links $O(nm)$
  3. output the dendogram of G

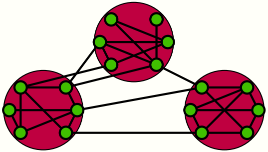
- Complexity
  - betweenness for all links : $O(nm)$
  - connected components : $O(m)$
  - m iterations
  - Overall : $O(nm^2)$

$O(n+m)$ $O(n^2)$

$n = 10^9$ $n = 10^5$

The Louvain algorithm

- Idea : optimize a quality function for node partitions



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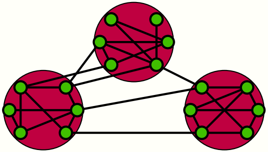

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- Problem... the best partition is a single community ! ! !

The Louvain algorithm

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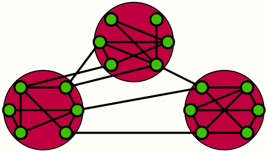

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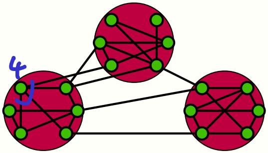
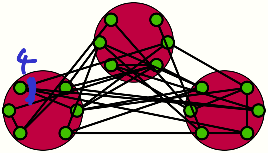
- Problem... the best partition is a single community ! ! !
- Correction : compare to a randomized version of the network



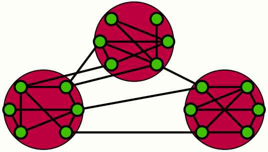
original network

$-$



configuration model

Modularity



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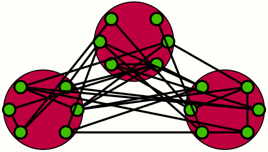
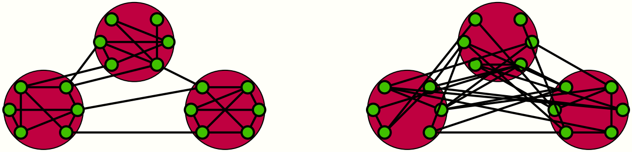


configuration model

- Proportion of edges inside communities

A the adjacency matrix of G
k_i the degree of node i
c_i the community of node i
δ is the Kronecker symbol : $\delta(c_i, c_j) = 1$ iff $c_i = c_j$
$= 0$ otherwise

Modularity



original network configuration model

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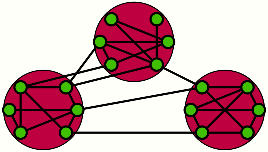
  - In the original network : $\frac{1}{2m} \sum_{i,j \in V} A_{ij} \delta(c_i, c_j)$ where

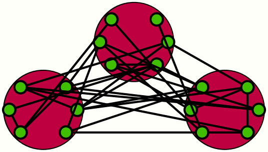
    (i, j)

Modularity

$k_i \in$ $\ni k_j$



original network



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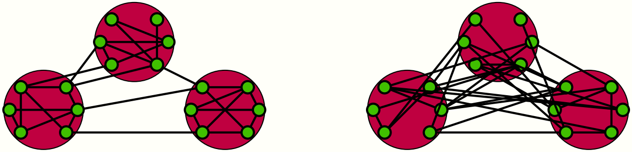
$\frac{k_i}{}$ $\frac{k_j}{}$

$2m$

$\frac{k_i k_j}{2m}$

Modularity



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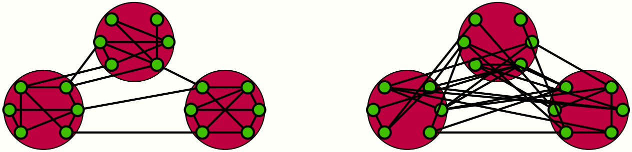
- **modularity** :

$$Q(\mathcal{P}) = \frac{1}{2m} \sum_{i,j \in V} [A_{ij} - \frac{k_i k_j}{2m}] \delta(c_i, c_j)$$
$$= \frac{1}{2m} \sum_{c \in \mathcal{P}} [e_c - \frac{a_c^2}{2m}]$$

number of demi links inside C

edges inside C

Modularity



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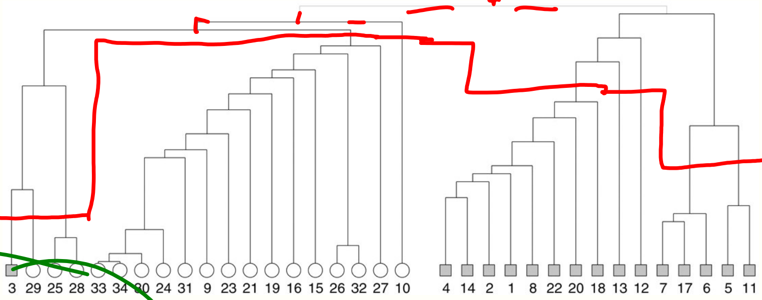
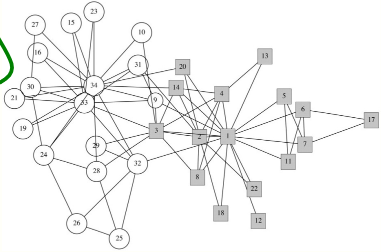
$$
\begin{aligned}
Q(\mathcal{P}) &= \frac{1}{2m}\sum_{i,j\in V}[A_{ij} - \frac{k_i k_j}{2m}]\delta(c_i, c_j) \\
&= \frac{1}{2m}\sum_{c\in\mathcal{P}}[e_c - \gamma\frac{a_c^2}{2m}]
\end{aligned}
$$

  - **NP-hard** to maximize modularity

Utility of modularity

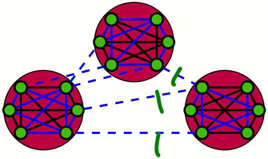
- Come back to the dendogram produced by Girvan-Newman





Other quality functions

- Distance to cluster graphs



  - dist-cluster($\mathcal{P}$)=#missing edges inside + #edges outside

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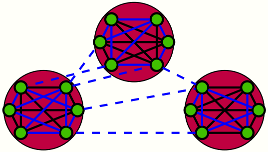

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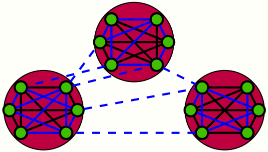

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- Constant Potts Model
  - CPM($\mathcal{P}$)=$\sum_c [e_c - \gamma \binom{n_c}{2}]$

    where e_c=# edges inside communauty c

    and n_c=# nodes in communauty c

    γ is a chosen constant ≤ 1

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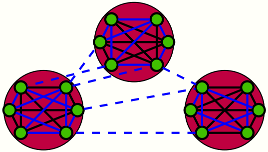

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  - for $\gamma = 0$?

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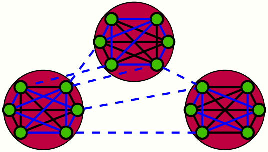

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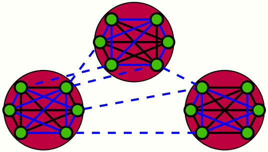
    where e_c=# edges inside communauty c
    and n_c=# nodes in communauty c
    γ is a chosen constant ≤ 1
  - for $\gamma = 0$? → all the nodes together
  - for $\gamma = 1$? → all the nodes isolated.

Other quality functions

- Distance to cluster graphs



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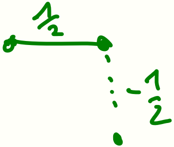
  - for $\gamma = 0$?
  - for $\gamma = 1$?
  - for $\gamma = 1/2$?



Is modularity a good quality function ?

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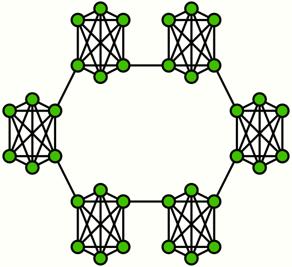
  Example : ring of p copies of a k-clique ($n = p.k$)



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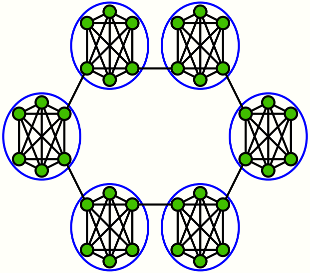

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Is modularity a good quality function ?

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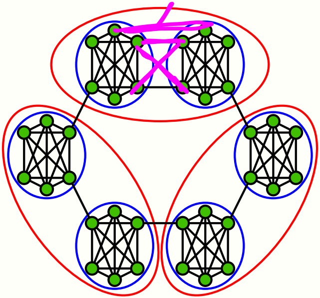
$+1$

$\frac{4 \times 4}{330}$



$p = 30$

$k = 5$

$\frac{25 \times 16}{330} = \frac{300}{330}$

$\color{blue}{\mathcal{P}_a}$ = the cliques
$\color{red}{\mathcal{P}_b}$ = the cliques grouped by two

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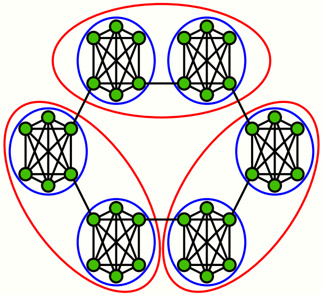

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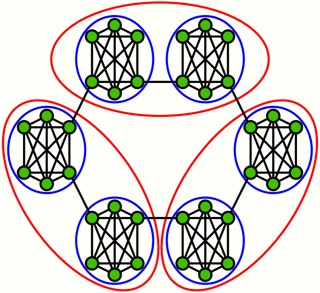

$\mathcal{P}_a$ = the cliques
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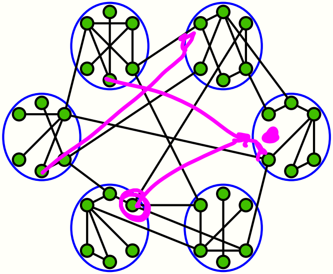
- Which one is "morally" the best community partition ?
- Which one has higher modularity ?

Louvain algorithm

- **Given a partition**, make a pass through all the vertices :
  - consider each vertex x once in an arbitrary order
  - move x to the community that gives the largest increase in modularity

G (n=30,m=46)



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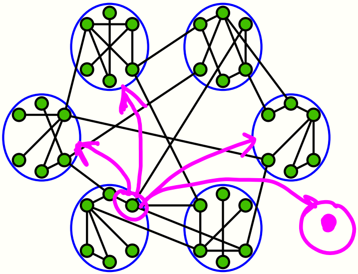

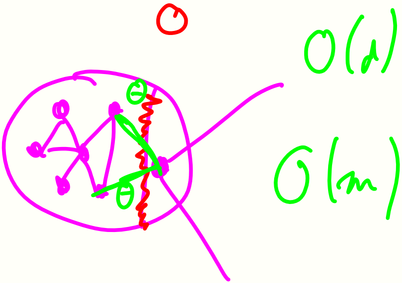

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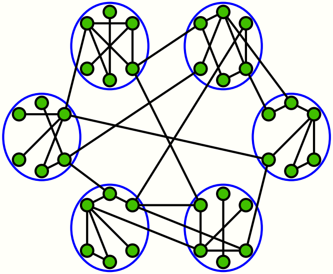

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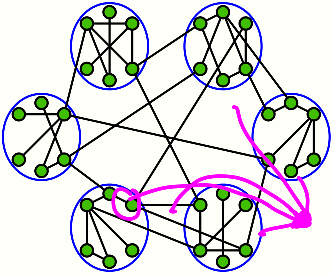

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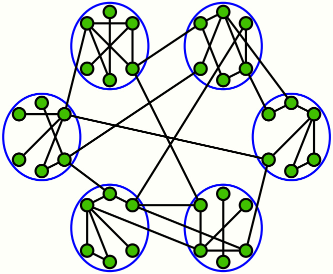

$O(1)$ time

$$\Delta Q(C,i) = \left[\frac{e_C + k_{i,C}}{2m} - \left(\frac{a_C + k_i}{2m}\right)^2\right] - \left[\frac{e_C}{2m} - \left(\frac{a_C}{2m}\right)^2 - \left(\frac{k_i}{2m}\right)^2\right]$$

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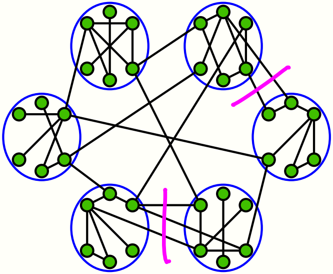
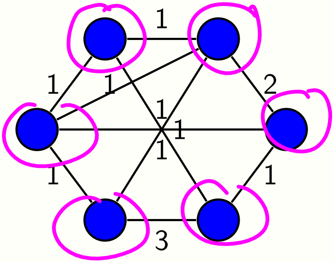


$G/\mathcal{P}$



Louvain algorithm

augmented$\leftarrow$ true;
while *augmented* **do**
$\mathcal{P}_0 \leftarrow \{\{x\} \mid x \in V(G)\}$; $\mathcal{P} \leftarrow \mathcal{P}_0$; $Q \leftarrow 0$;
while *augmented* **do**
augmented$\leftarrow$ faux;
for *i de 1 a n* **do**
$Q_{ori} \leftarrow Q$;
i moves to $c_{iso} = \{i\}$; $Q \leftarrow Q - \Delta Q_{out}(i)$;
$Q_{max} \leftarrow Q$; $c_{max} \leftarrow c_{iso}$;
for $c \in \mathcal{P}$ **do**
if $Q + \Delta Q_{in}(c) > Q_{max}$ **then**
$Q_{max} \leftarrow Q + \Delta Q_{in}(i, c)$;
$c_{max} \leftarrow c$;
tas
end
end
If $Q_{max} = Q_{ori}$ **then** $c_{max} \leftarrow c_{ori}$ **else** augmented$\leftarrow$ true;
i moves to c_{max} ; $Q \leftarrow Q_{max}$;
end
end
If $\mathcal{P} \neq \mathcal{P}_0$ **then** augmented $\leftarrow$ true ; $G \leftarrow G/\mathcal{P}$;
end
return $\{Expand(P) \mid P \in \mathcal{P}\}$;

Leiden algorithm

Two improvements over Louvain

- Complexity

Leiden algorithm

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  - Consider moving only vertices whose neighbours have moved

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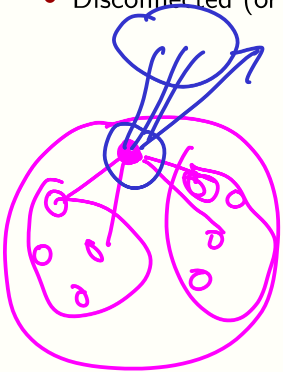

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  - Contract only the obtained sub-communities

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- Disconnected (or poorly connected) communities
  - Just before contracting communities, for each community
    - Place vertices alone in their own sub-community
    - Merge sub-communities that are strongly connected
  - Contract only the obtained sub-communities
  - At the next step start from the partition defined by the whole communities