SPATIAL NETWORKS

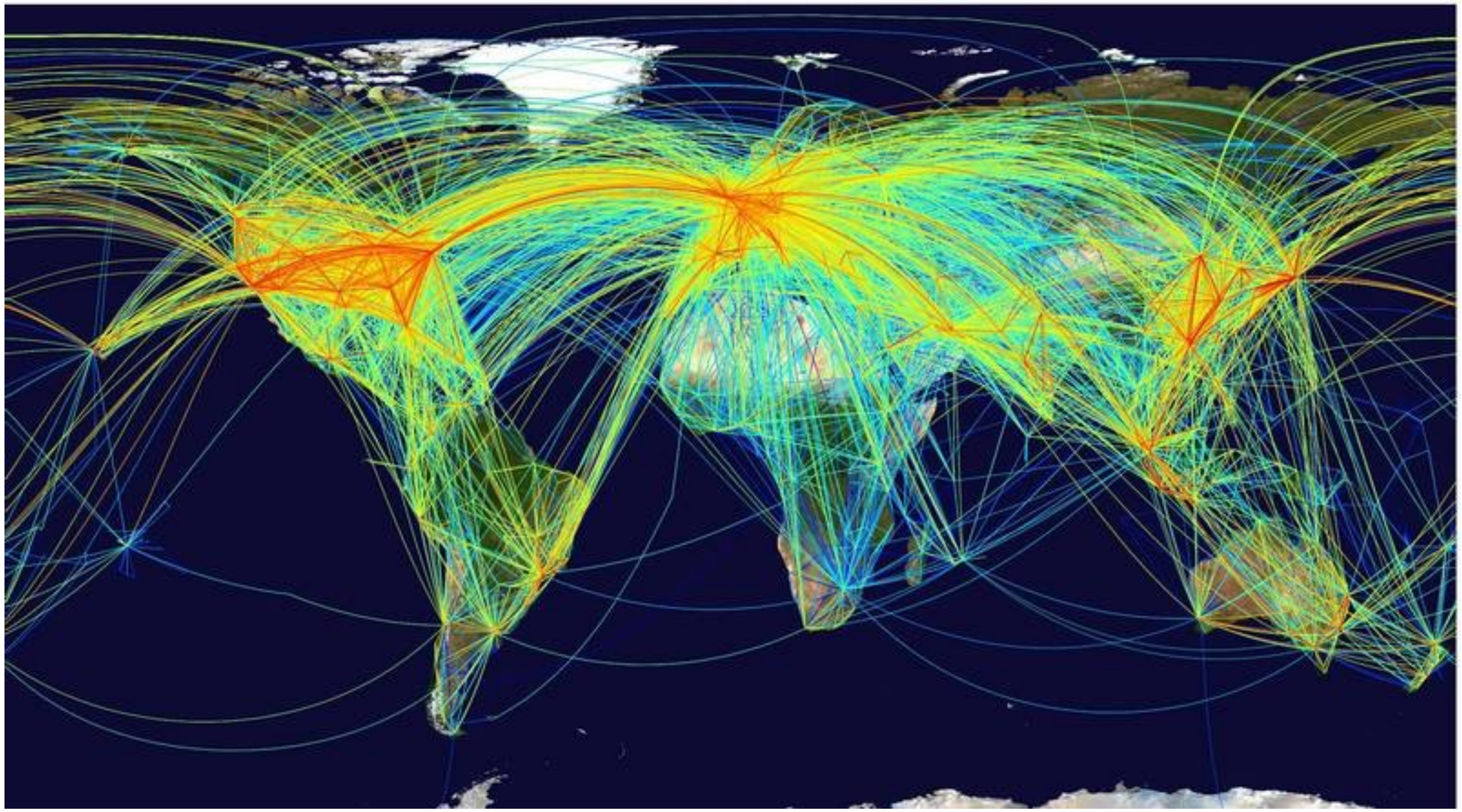
Spatial networks

A network is said spatial if the distance between nodes affect the probability of observing edges between them

- Can be seen as a generalisation of Assortativity

Distance

- Physical distance
- Economical distance
- Social distance
- Difference in professional categories
- …



Geisel et.al. (2011)

Literature

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Spatial networks

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A R T I C L E   I N F O

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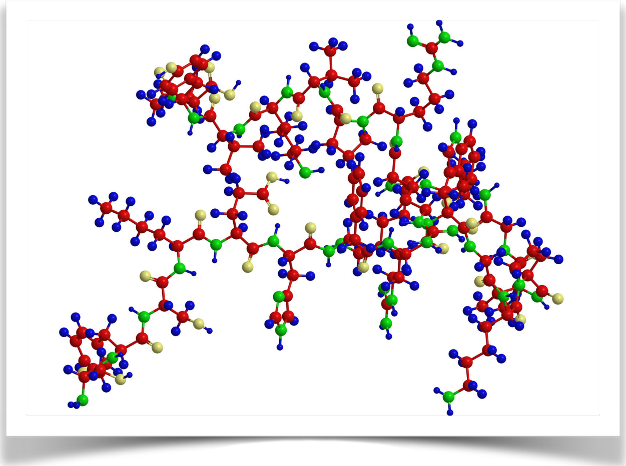
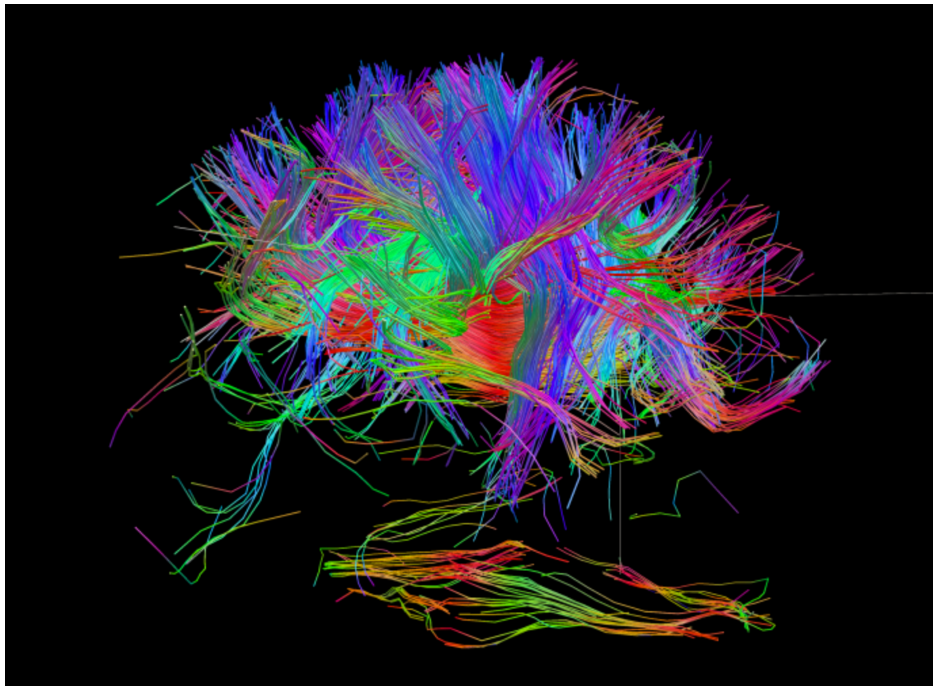
Complex systems are very often organized under the form of networks where nodes and edges are embedded in space. Transportation and mobility networks, Internet, mobile phone networks, power grids, social and contact networks, and neural networks, are all examples where space is relevant and where topology alone does not contain all the information. Characterizing and understanding the structure and the evolution of spatial networks is thus crucial for many different fields, ranging from urbanism to epidemiology.

Spatial networks

Types of spatial networks

- Transportation networks
  - Airline networks
  - Bus, subway, railway, and commuters
  - Cargo ship networks
- Infrastructure networks
  - Road and street networks
  - Power grids and water distribution networks
  - The internet
- Neural networks
- Protein networks
- Mobility networks
- Social networks
- …

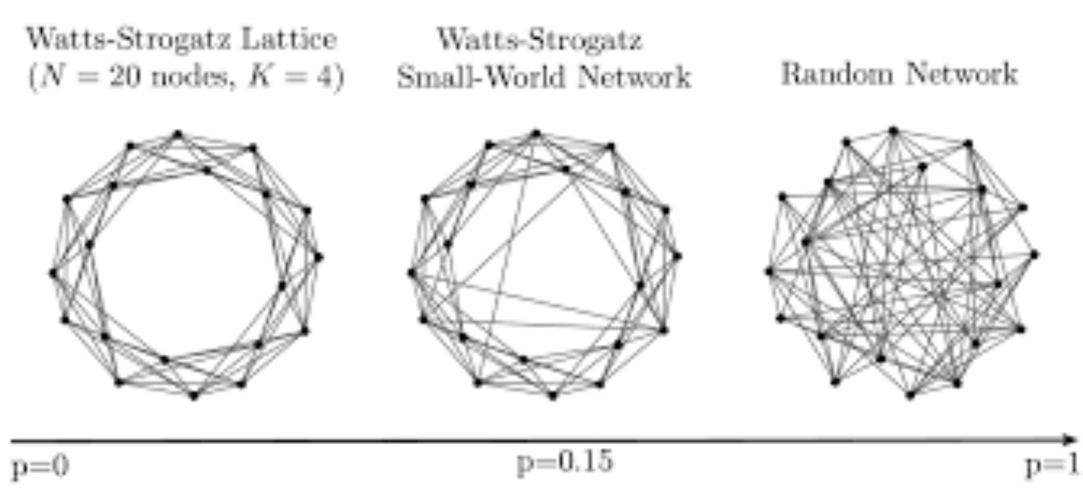


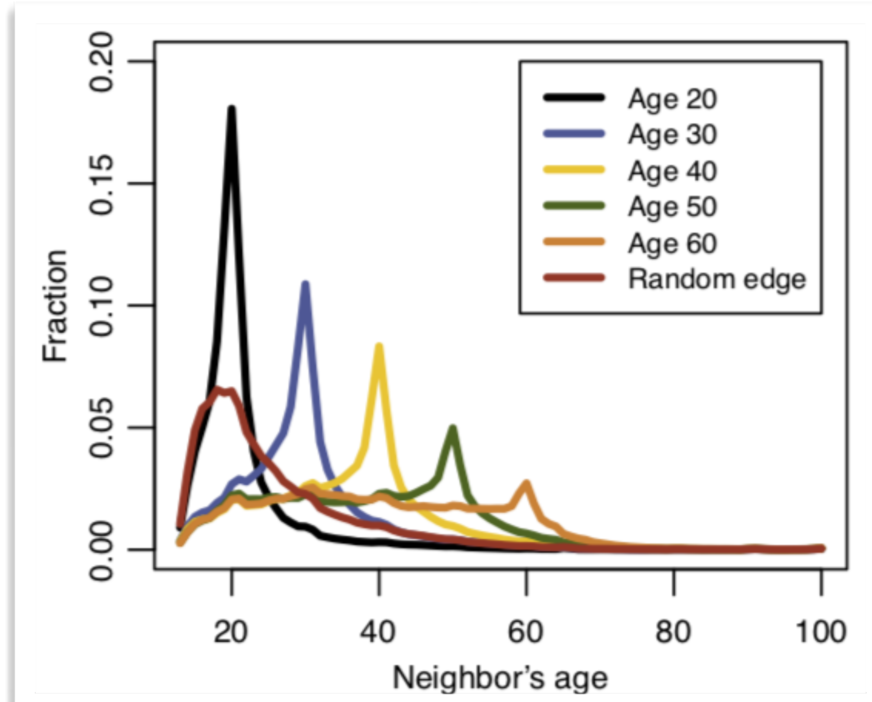


Spatial networks

Examples of 1D spaces

- The watts-Strogatz random graph is defined on a (circular) 1D space: each node is (initially) connected to its k closest nodes in this space.
- In social networks, users tend to be more connected with other users with similar age. We can consider age as a position on 1D space. The same is true about political opinions, if we consider a Left-Right spectrum.

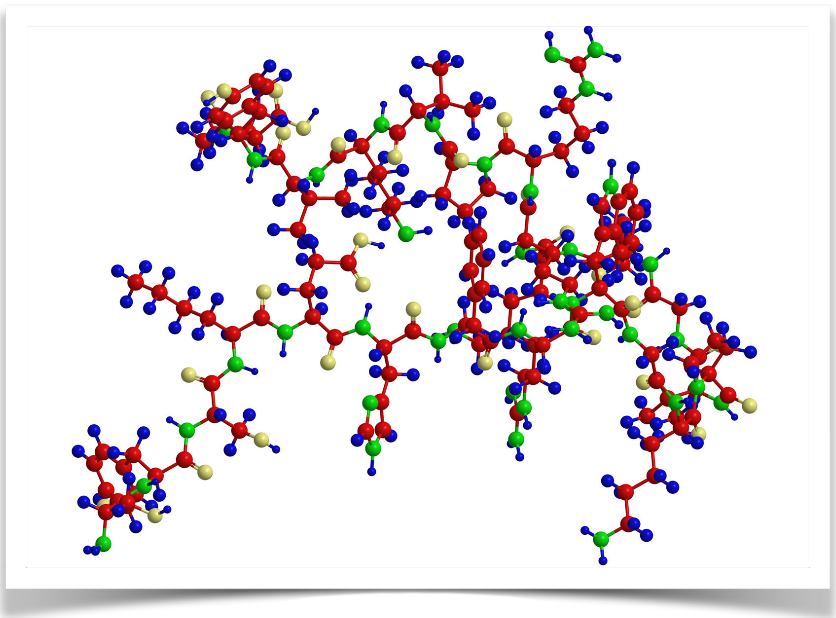


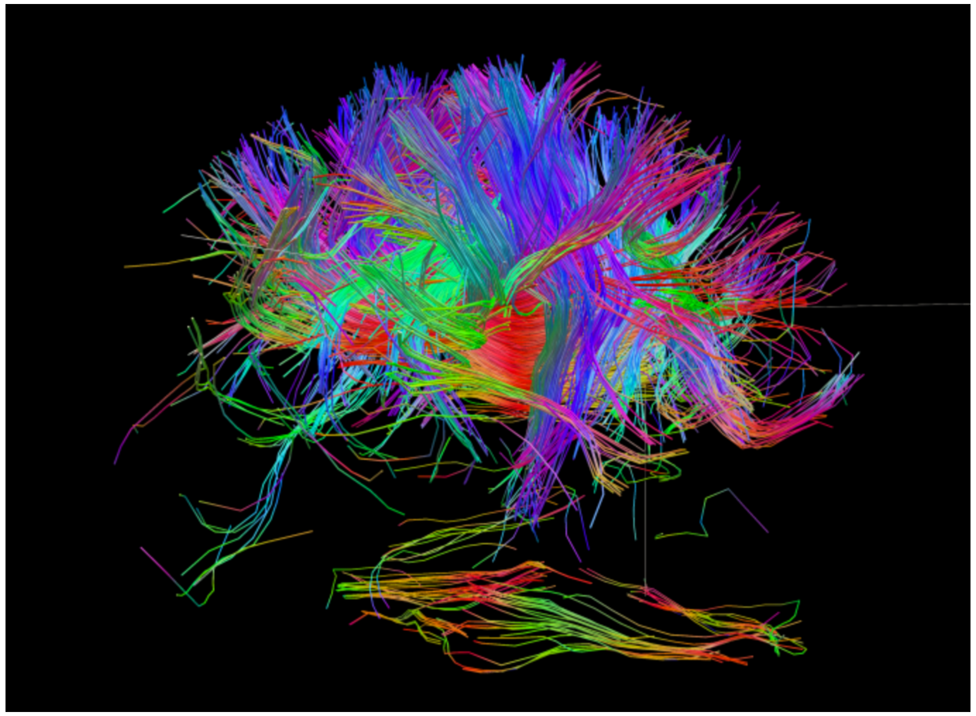


Spatial networks

Examples of 3+D spaces

- If we consider altitude, geographical networks are 3D spaces
- If we consider multiple nodes properties as dimensions, nodes can be located on high dimensional spaces, e.g., age, political opinion, revenue, geographical location, etc. Be careful however, that analyzing a spatial networks needs to define the *distance* between nodes, which can be tricky to define if dimensions are of different natures.
- Methods such as *graph embedding* assign locations in arbitrary large dimensions to nodes that summarize some of the network properties (see later class).





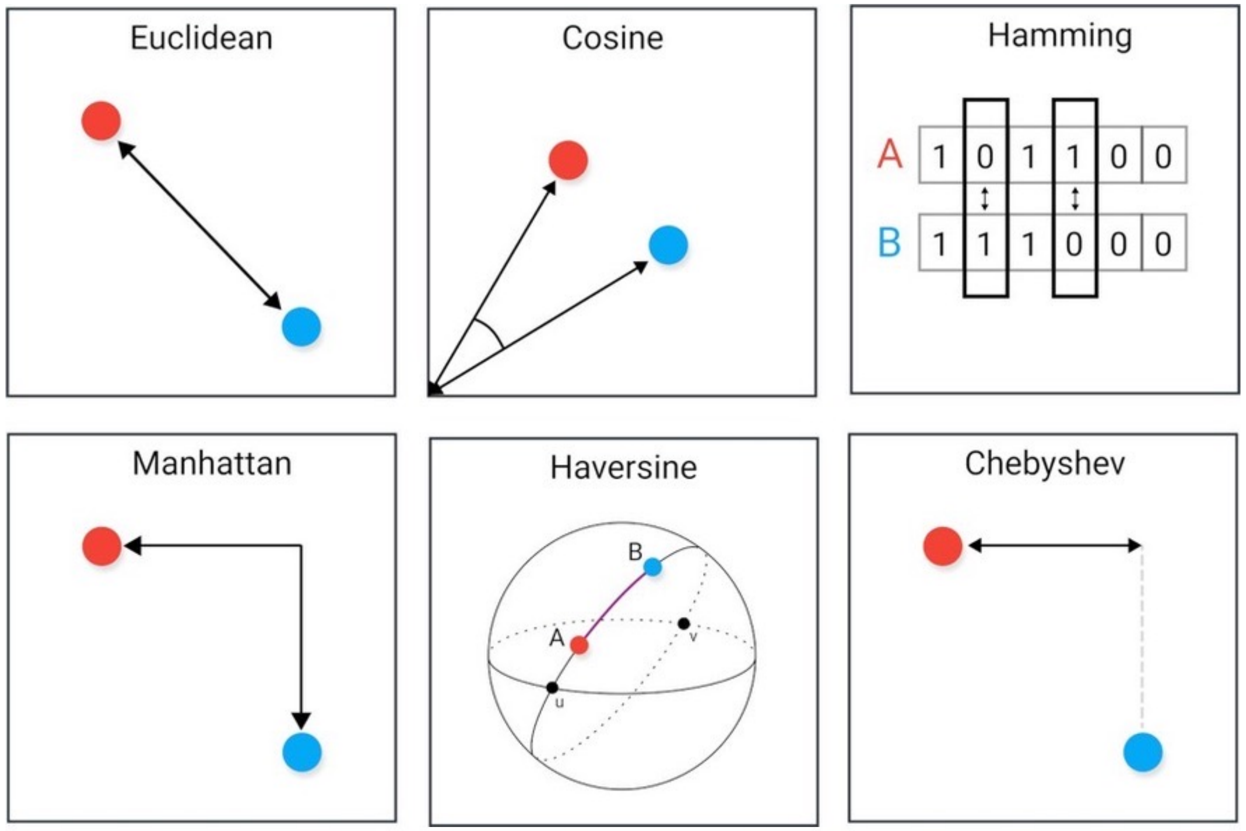
Spatial networks

Distances

The distance between each pair of nodes can be computed in different ways, depending on the nature of dimensions nodes are embedded in. The most common ones are:

- **Euclidean distance**, or $L^2 distance$ is the usual, straight line distance
- **Great-Circle distance** is used to measure the distance between points located on a sphere, typically the Earth for geographical data.
- **Dot product** and **Cosine Distance** are often used in high dimensions, in particular when it makes sense to multiply the location vectors.
- **Manhattan distance**, or $L^1 distance$, is sometimes used as a variant of Euclidean distance for high dimensional data (it is simply defined as the sum of differences in each of the dimensions.)
- Observed distance can sometimes be used, a typical example being **average time distance**: in datasets of trips or traffic, the time distance between dots might be only loosely proportional to geographical distance.

Spatial networks



(Chess distance)

https://towardsdatascience.com/9-distance-measures-in-data-science-918109d069fa

Spatial networks

Notation

Δ_{uv}	**Metric distance** between u and v (Euclidean, Manhattan, etc.)
ℓ_{uv}	**Route distance** between u and v, i.e., sum of Metric distances between nodes on the shortest path between u and v

Spatial networks

Route factor - Accessibility

$Q(u,v)$ **Route Factor**, also called the detour index, measures how *efficiently* the network allows to go from a node to another, it is defined as the ratio between the metric distance and the route distance:

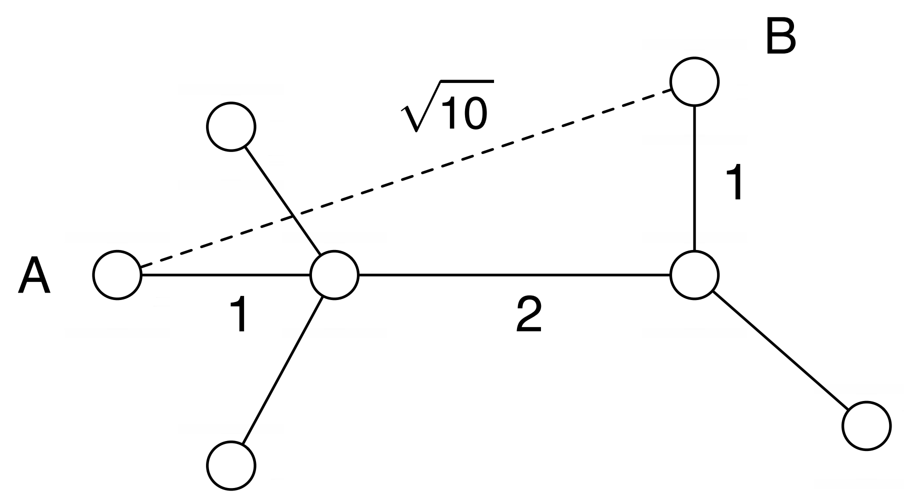
$$Q(u,v) = \frac{\Delta_{uv}}{\ell_{uv}}$$

$\langle Q(u) \rangle$ **Node Accessibility**: Average route factor from a node to all others:

$$\langle Q(u) \rangle = \frac{1}{N-1} \sum_{v} Q(u,v)$$

$\langle Q \rangle$ **Accessibility**: Average route factor for the whole network:

$$\langle Q \rangle = \frac{1}{N(N-1)} \sum_{u \neq v} Q(u,v)$$



Simple models of spatial networks

Random geometric graphs

General definition:

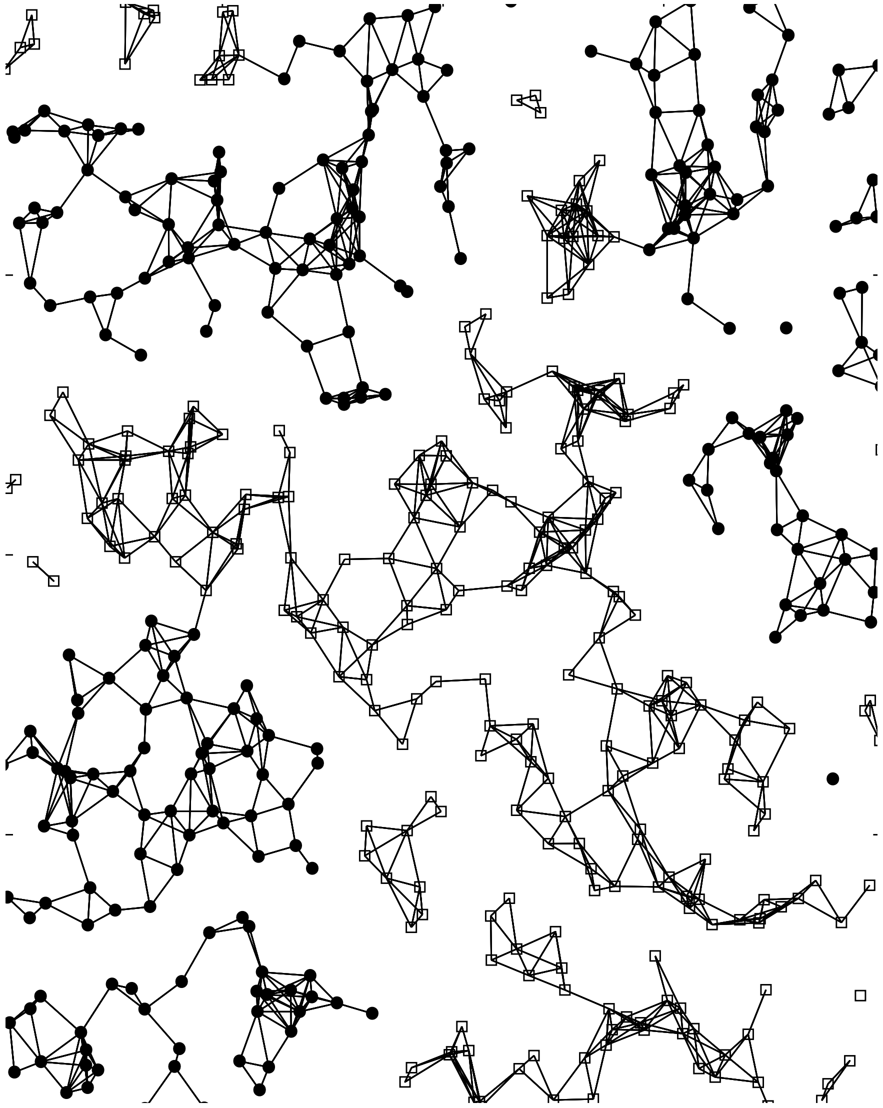
- Take a space and distribute nodes randomly
- Nodes are small spheres with radius r
- Two nodes are connected if their spheres overlap — separated with distance smaller than 2r
- Also called: disk-percolation

Degree distribution — Poisson distribution

Clustering coefficient (d=dimensions)

$$\langle C_d \rangle \sim 3\sqrt{\frac{2}{\pi d}}\left(\frac{3}{4}\right)^{\frac{d+1}{2}}$$

Independent of N contrary to random networks



Jesper Dall and Michael Christensen. “Random geometric graphs”. In: *Physical review E* 66.1 (2002), p. 016121.

Soft RGG

Soft RGG (Waxman random graph)

Soft RGG, or **Waxman Random Graphs**[a], starts as the RGG by distributing nodes at random in a space, but instead of adding links between all nodes closer than a certain distance, it assign edges between nodes according to a **deterrence function** f, i.e., a function defining how distance affects the probability of observing edges between nodes.

The Soft RGG can model an ER random graph if f is constant function, $f(\Delta) = p$. It can model a classic RGG if f is a threshold function with:

$$f(d) = \begin{cases} 1 & \Delta \leq r \\ 0 & \Delta > r \end{cases}$$

[a] Waxman 1988.

Deterrence function

Deterrence function

A deterrence function defines how the distance affects the probability of observing an edge. It can be a probability (bounded on $[0, 1]$), or define a change ratio.

1. It can be defined *a priori*, usually as a classic monotonically decreasing function, e.g., Negative exponential($f(\Delta) = e^{-\alpha\Delta}$) or Negative power ($f(\Delta) = \Delta^{-\alpha}$), with α a parameter. A typical example of negative power in geographical data is when the probability of observing an edge decreases as the square of the distance, i.e., $f(\Delta) = \frac{1}{\Delta^2}$
2. It can also be learned from data, either by fitting parameters of a predefined function (e.g., the α parameter above), or by using an *Ad-Hoc deterrence function*.

The gravity law

Formal description

Origin-destination matrix

- Describe flow of individuals between locations
- Used since decades by geographers
- Definition:
  - divide the area of interest into zones (cells) labelled by $i=1...N$
  - count the number of individuals going from location i to location j

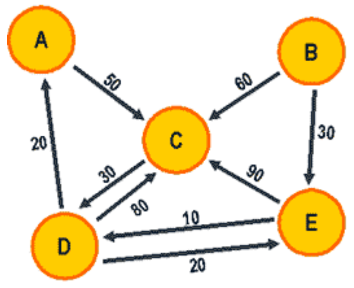
- directed
- weighted
- Beware:
  - strongly depends on the zone definition

O/D Matrix

$T(i,j)=$

	A	B	C	D	E	Ti
A	0	0	50	0	0	50
B	0	0	60	0	30	90
C	0	0	0	30	0	30
D	20	0	80	0	20	120
E	0	0	90	10	0	100
Tj	20	0	280	40	50	390

Spatial Interactions



The gravity law

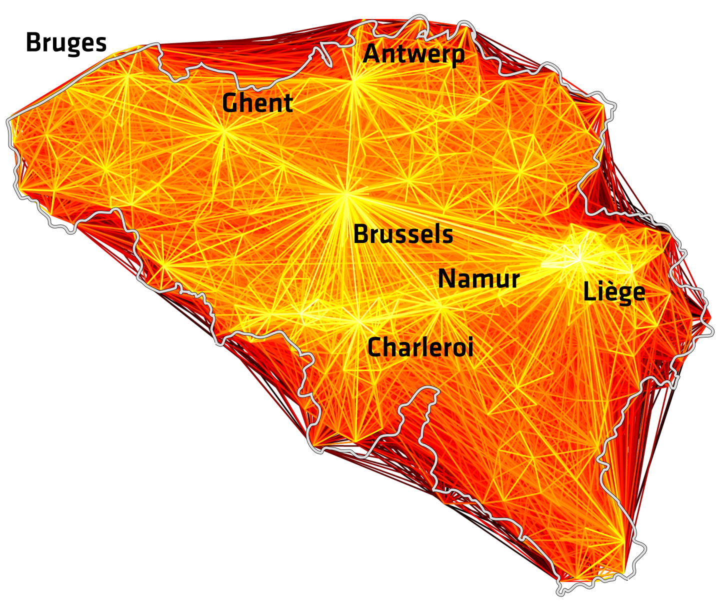
Number of trips from location i to location j is scaling as

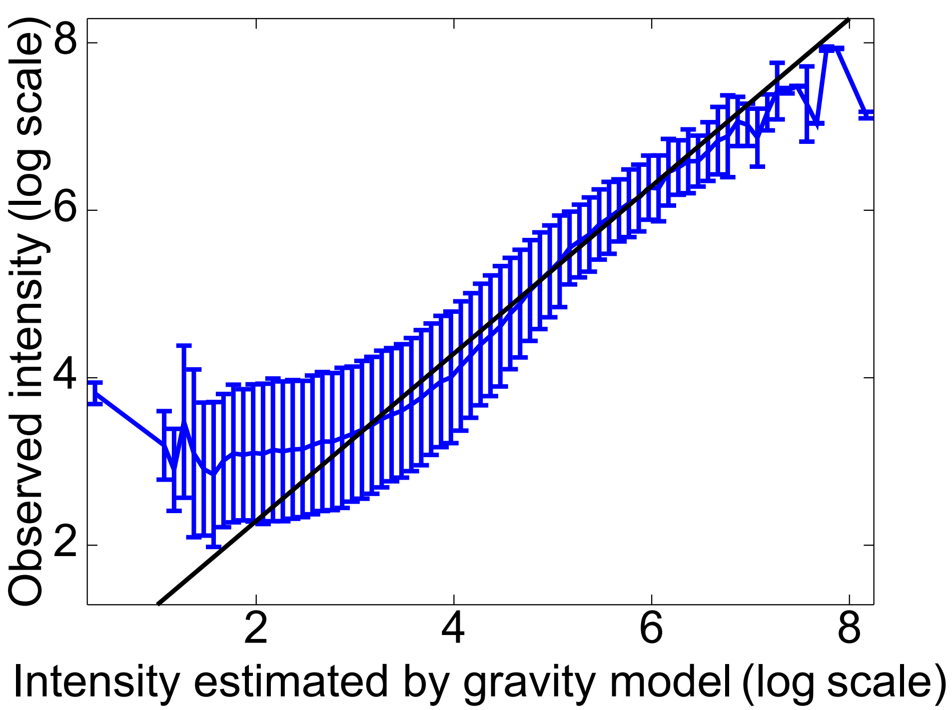
$$T_{ij} = K \frac{P_i P_j}{d_{ij}^{\sigma}}$$

- where $d_{ij} = d_E(i,j)$ is the distance between i and j
- $P_{i(j)}$ is the *population size* at location $I(j)$
- σ a parameter chosen or learned from data

Inter-city phone communication (Krings et.al.)

- mobile call communication intensity between Belgian cities





The gravity law - empirical summary

Both exponential and power-law dependence is observable

Network [Ref.]	N	Gravity law form	Results
Railway express [164]	13	$P_i P_j / d_{ij}^{\sigma}$	$\sigma = 1.0$
Korean highways [161]	238	$P_i P_j / d_{ij}^{\sigma}$	$\sigma = 2.0$
Global cargo ship [104]	951	$O_i I_j d_{ij}^{-\sigma} \exp(-d_{ij}/\kappa)$	$\sigma = 0.59$
Commuters (worldwide) [162]	n/a	$P_i^{\alpha} P_j^{\gamma} \exp(-d_{ij}/\kappa)$	$(\alpha, \gamma) = (0.46, 0.64)$ for $d < 300$ km
			$(\alpha, \gamma) = (0.35, 0.37)$ for $d > 300$ km
US commuters by county [163]	3109	$P_i^{\alpha} P_j^{\gamma} / d_{ij}^{\sigma}$	$(\alpha, \gamma, \sigma) = (0.30, 0.64, 3.05)$ for $d < 119$ km
			$(\alpha, \gamma, \sigma) = (0.24, 0.14, 0.29)$ for $d > 119$ km
Telecommunication flow [134]	571	$P_i P_j d_{ij}^{-\sigma}$	$\sigma = 2.0$

The gravity law

Number of trips from location i to location j is scaling as

$$T_{ij} = K\frac{P_i P_j}{d_{ij}^{\sigma}}$$

- where $d_{ij} = d_E(i,j)$ is the distance between i and j
- $P_{i(j)}$ is the *population size* at location $i(j)$

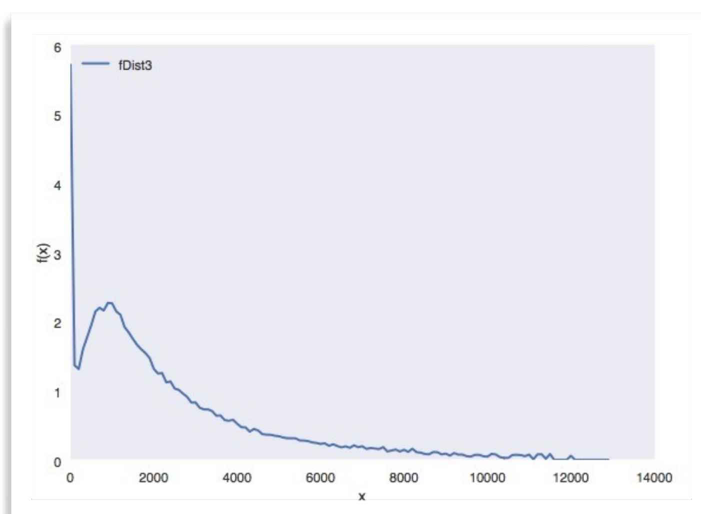
- In a general form: $T_{ij} \sim P_i P_j f(d(i,j))$
  - where $f(d(i,j))$ is the deterrence function describing the effect of space

Ad-hoc deterrence function

Agnostic deterrence function

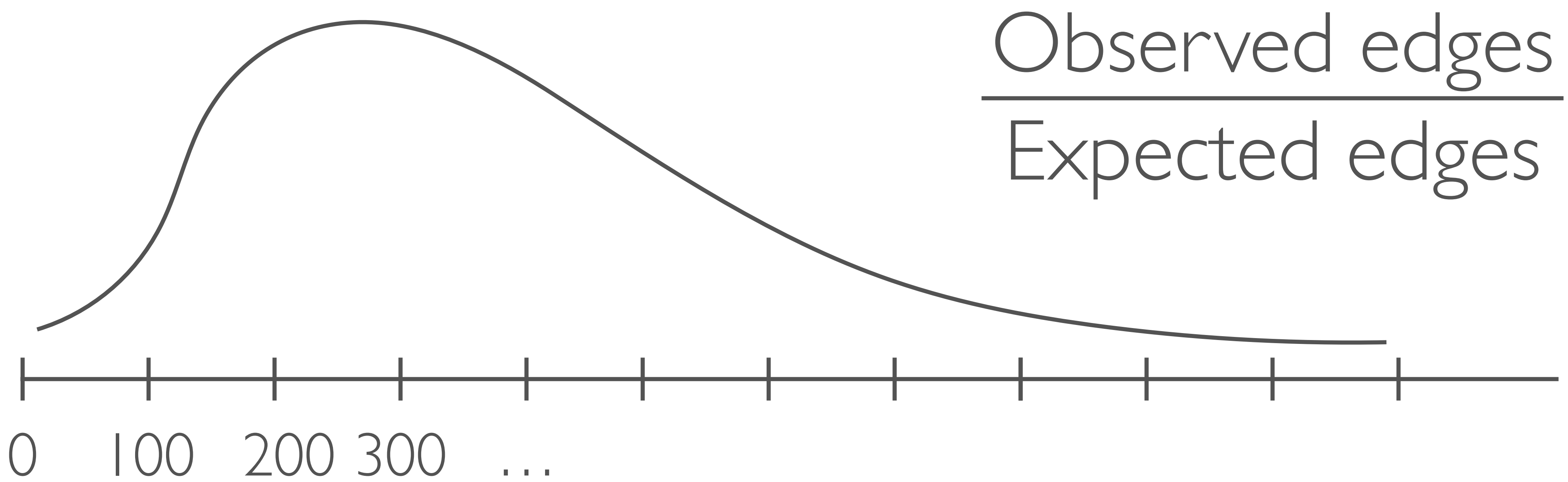
- The influence of distance might be more complex than a power-law or an exponential. In particular, it is often non-monotonic (first increasing, then decreasing. Think of airplanes, bicyles, public transports… unlikely to use for short distances)
- A deterrence function can be learned from data
- Computed by comparing the number of trips observed at a given distance with the number of trip expected if distance has no effect (a configuration model)

$f(d)$



Distance d

Ad-hoc deterrence function



$$f(d) = \frac{\sum_{i,j|\Delta_{ij}=d} A_{ij}}{\sum_{i,j|\Delta_{ij}=d} M_{ij}}$$

with A_{ij} the adjacency matrix (or weight matrix) of the observed graph and M_{ij} the probability of observing an edge (or weight of edges) between nodes i and j according to the chosen null model. For instance, with the simplest hypothesis that edges occur completely at random, $\forall_{i,j}, M_{ij} = d$.

The gravity law - as a network null model

Usage as a **network** null model

- Consider a spatial network (e.g., phone calls, trips, etc.)

- Fit a gravity model best explaining the observed network. If the population is unknown or not relevant, the degrees of nodes (in/out degrees in directed networks) can be used as a "*population*"

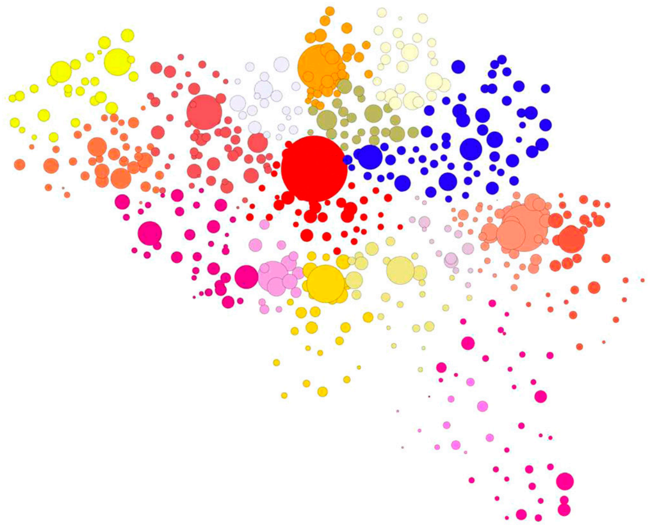
- =>Random model with a given edge probability for each pair of node

- The obtained network is a null model to which the observed network can be compared

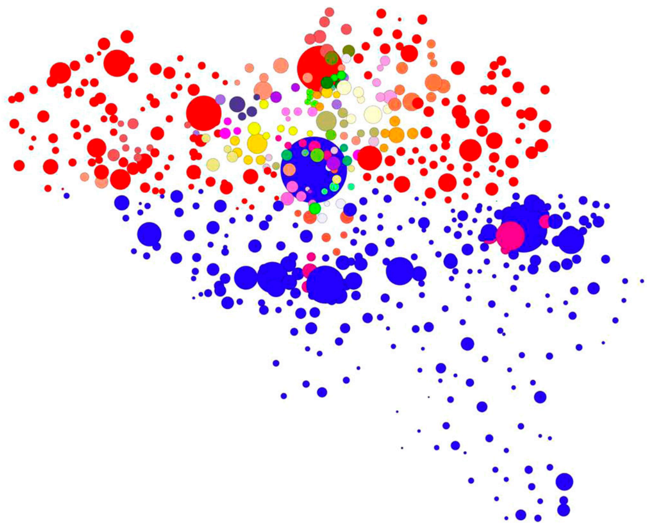
The gravity law - as a network null model

Example of application: Space-independent communities

- In the usual modularity, the fraction of internal edges is compared between the observed network and a configuration model.
- One can replace the configuration model by a gravity model



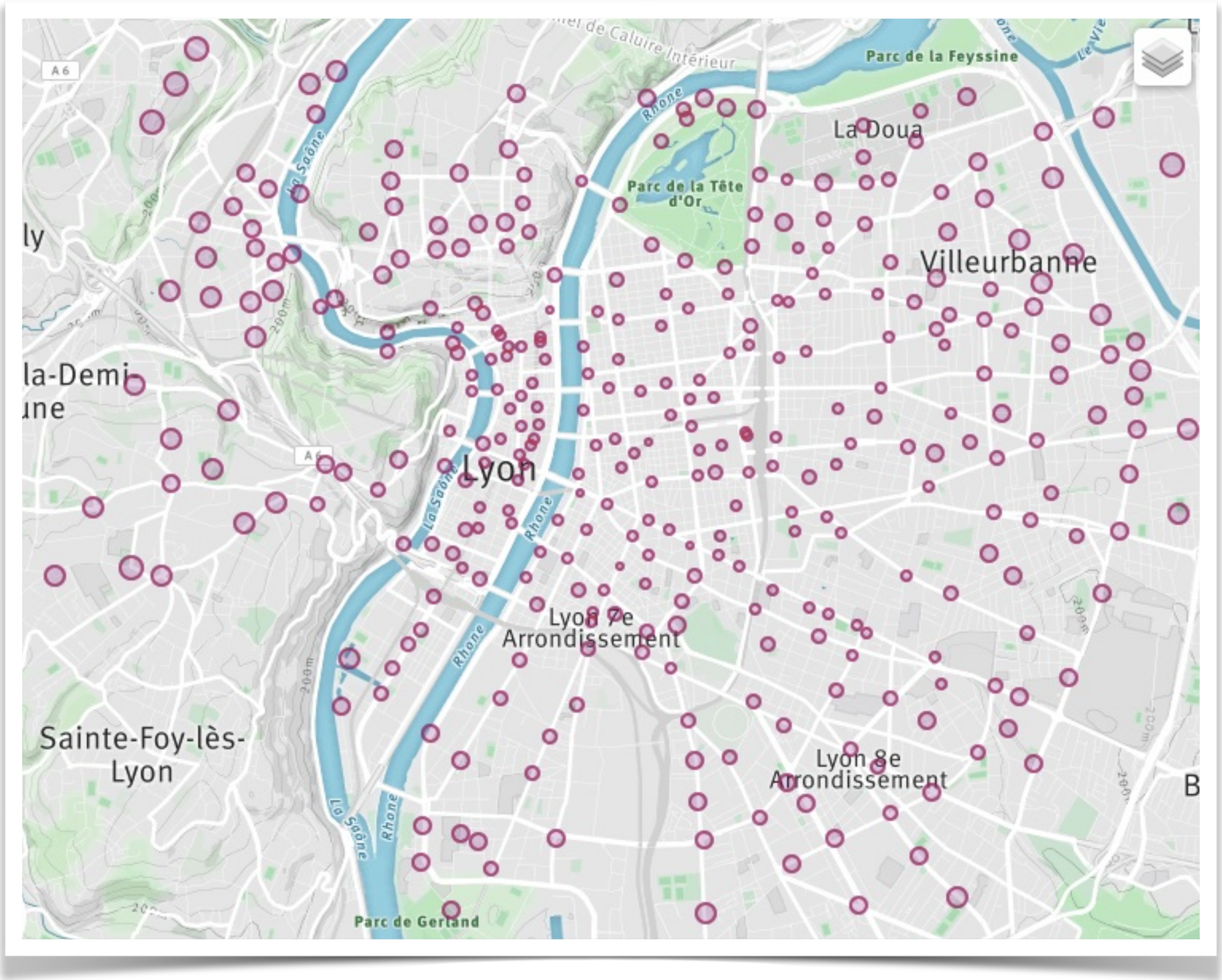
Space-dependent communities



Space-independent communities

Expert, P., Evans, T. S., Blondel, V. D., & Lambiotte, R. (2011). Uncovering space-independent communities in spatial networks. *Proceedings of the National Academy of Sciences*, *108*(19), 7663-7668.

The gravity law - example

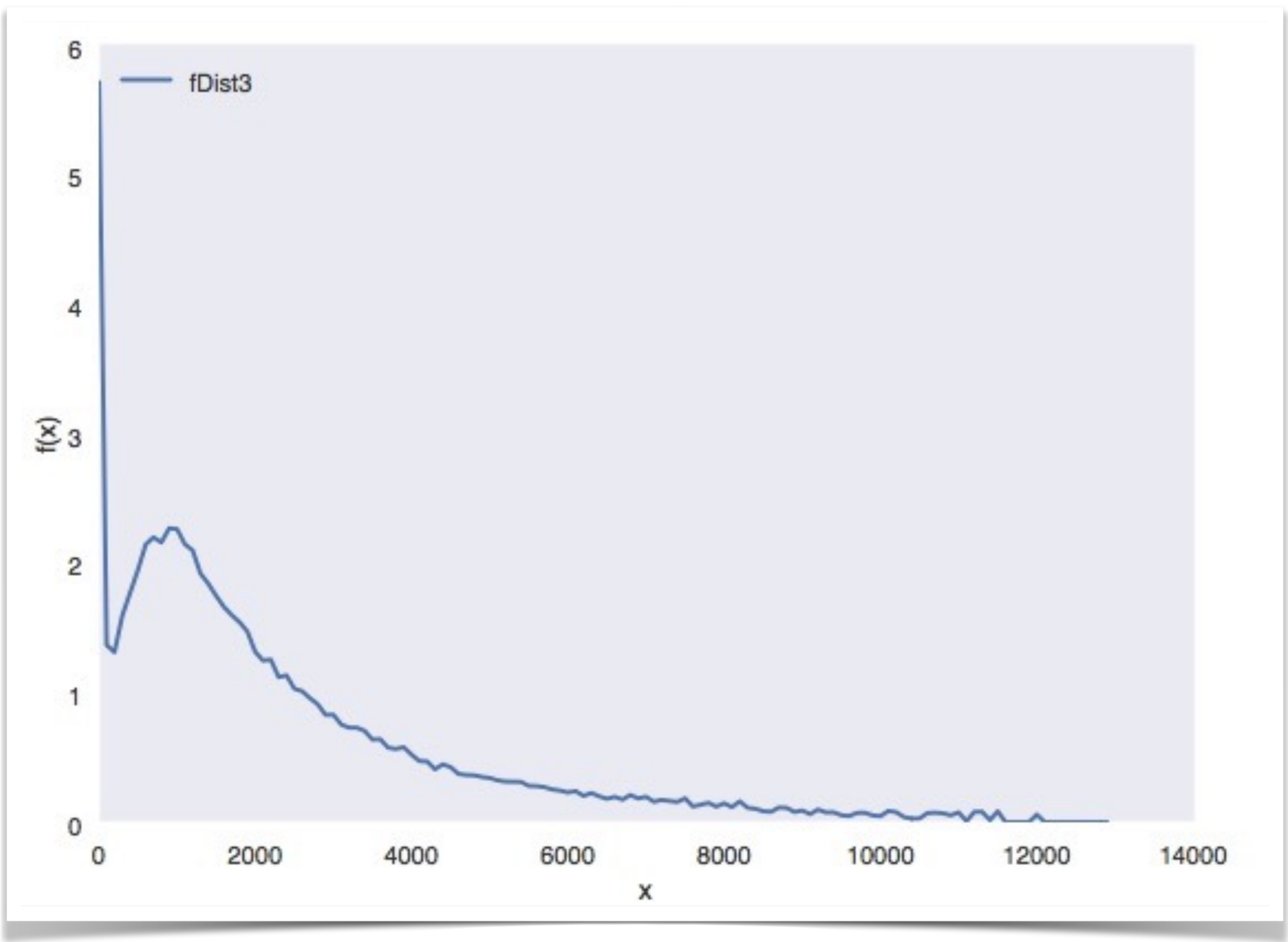


Nodes: Vélo'v station (2D position)

Edges: number of trips over a period

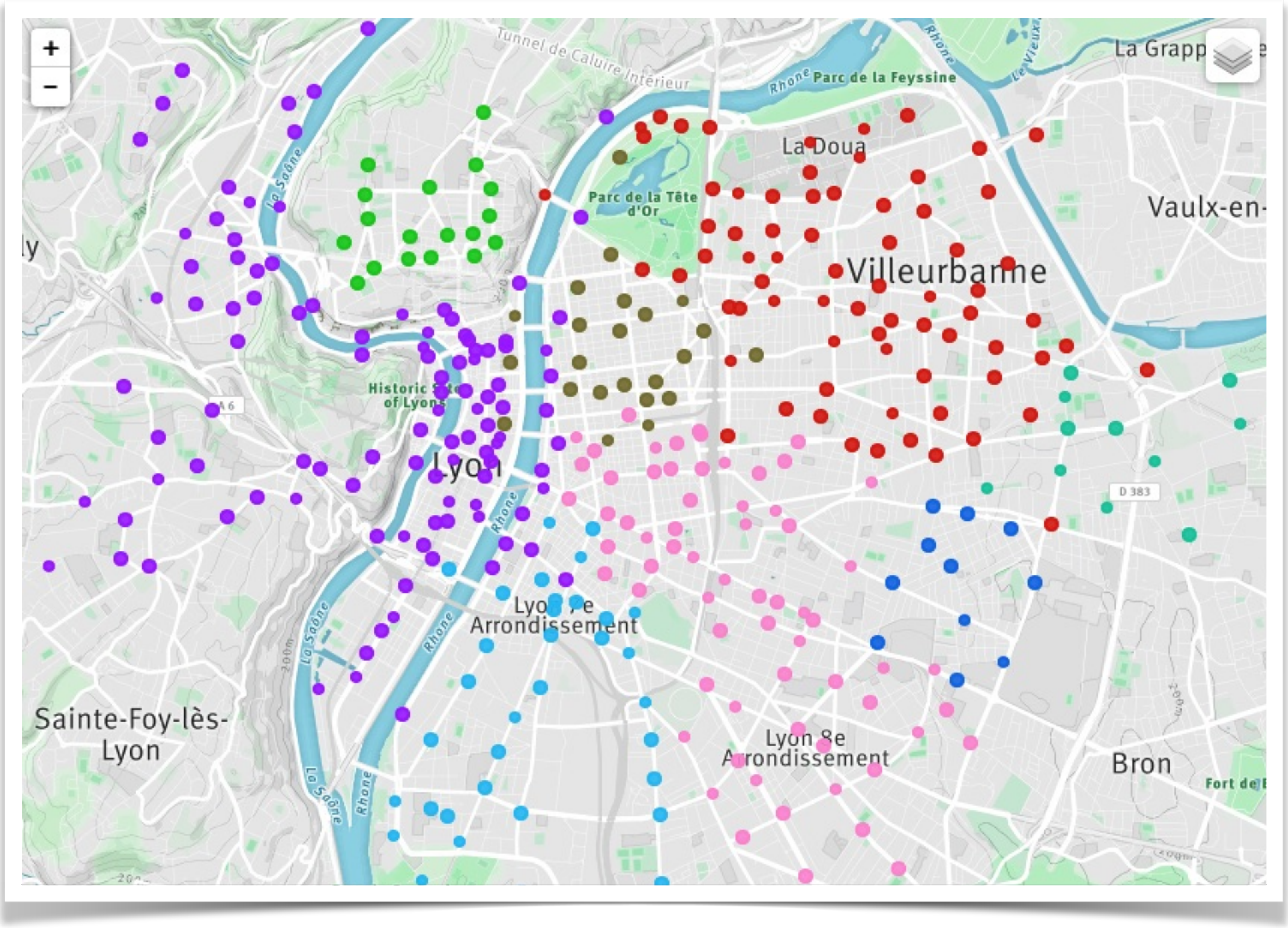
The gravity law - example

$f(d)$



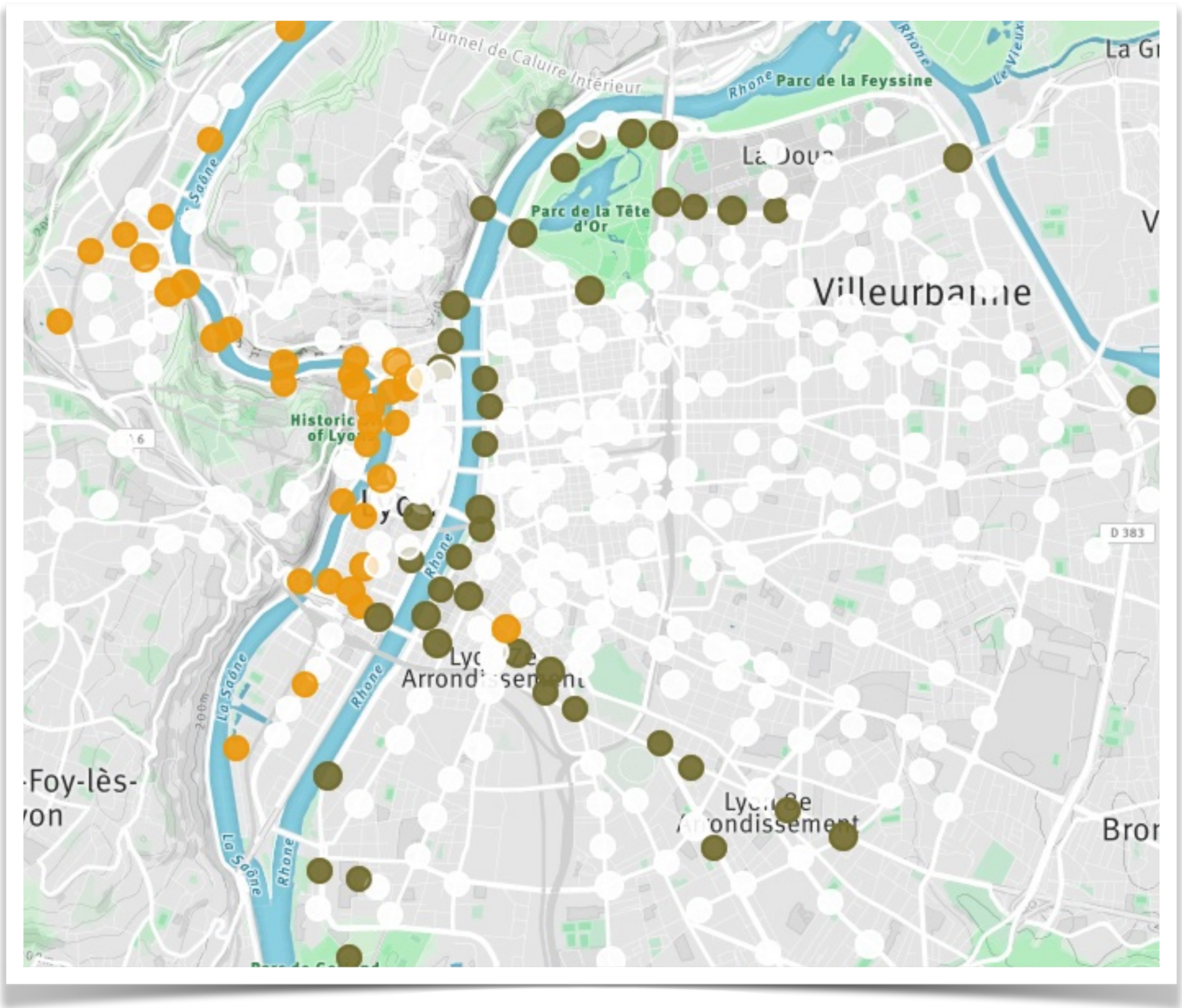
Distance *d* *(meters)*

The gravity law - example



Space-dependent communities

The gravity law - example



Some (social) space-independent communities that were previously *hidden* by spatial constraints

The radiation law

The radiation law

Limitations of the gravity law

1. Requires previous data to fit

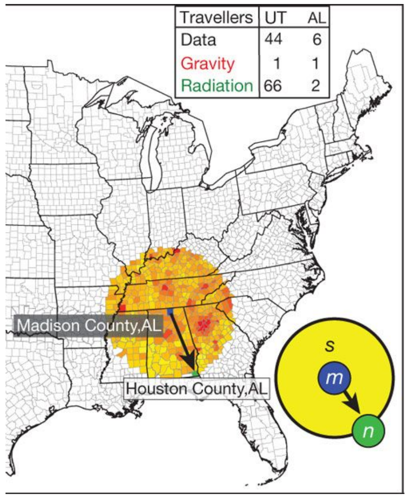
2. The number of travelers between destinations depends only on their populations and distances. In reality, this value depends probably of other *opportunities*

The radiation law

Intuition: Model how people move for jobs

1. Individuals look for job in all cities
2. Each city has a number of job opportunities
   - Each job has a value of *interest*, considered random
3. What is the probability for a job-seeker to choose a job in city *c* located at distance *d?*
   - Depends only on how many jobs offered in cities at a distance equal or lower than *d* (probability to find a better job closer)

The model is parameter-free!



The radiation law

The model can be formulated in terms of radiation and absorption

- take locations i and j with populations (in-degree) m_i and n_j and at distance r_{ij}
- denote s_{ij} the total population in the circle with radius r_{ij} centered at i (excluding the source and destination population)
- P is the *power of attraction*, I.e., without other data, the degree.

Radiation Law of Spatial Interactions

The **Radiation Law**[a] is another random spatial model. Unlike previous ones, it does not depends on a deterrence function, and is parameter-free. It is based on the principle of relative opportunities: the probability of observing an interaction from i to j depends on P_i^{out}, P_j^{in}, and the sum of all P_k^{in} for $\Delta_{ik} < \Delta_{ik}$, i.e., other opportunities accessible at a shorter distance. More formally:

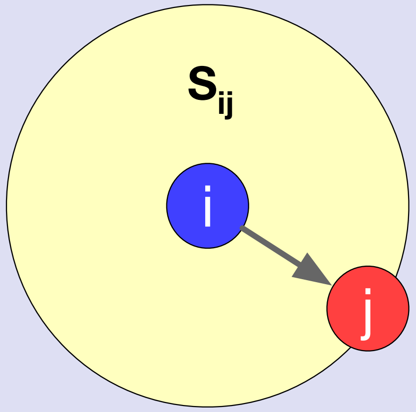
$$R_{ij} = k_i^{out} \frac{P_i^{out} P_j^{in}}{(P_i^{out} + s_{ij})(P_i^{out} + P_j^{in} + s_{ij})}$$

With $s_{ij} = \sum_{u \in V, \Delta_{iu} < \Delta_{ij}} P_u^{in}$ the sum of opportunities at a shorter distance than the target.

[a] Simini et al. 2012.

Radiation Law of Spatial Interactions

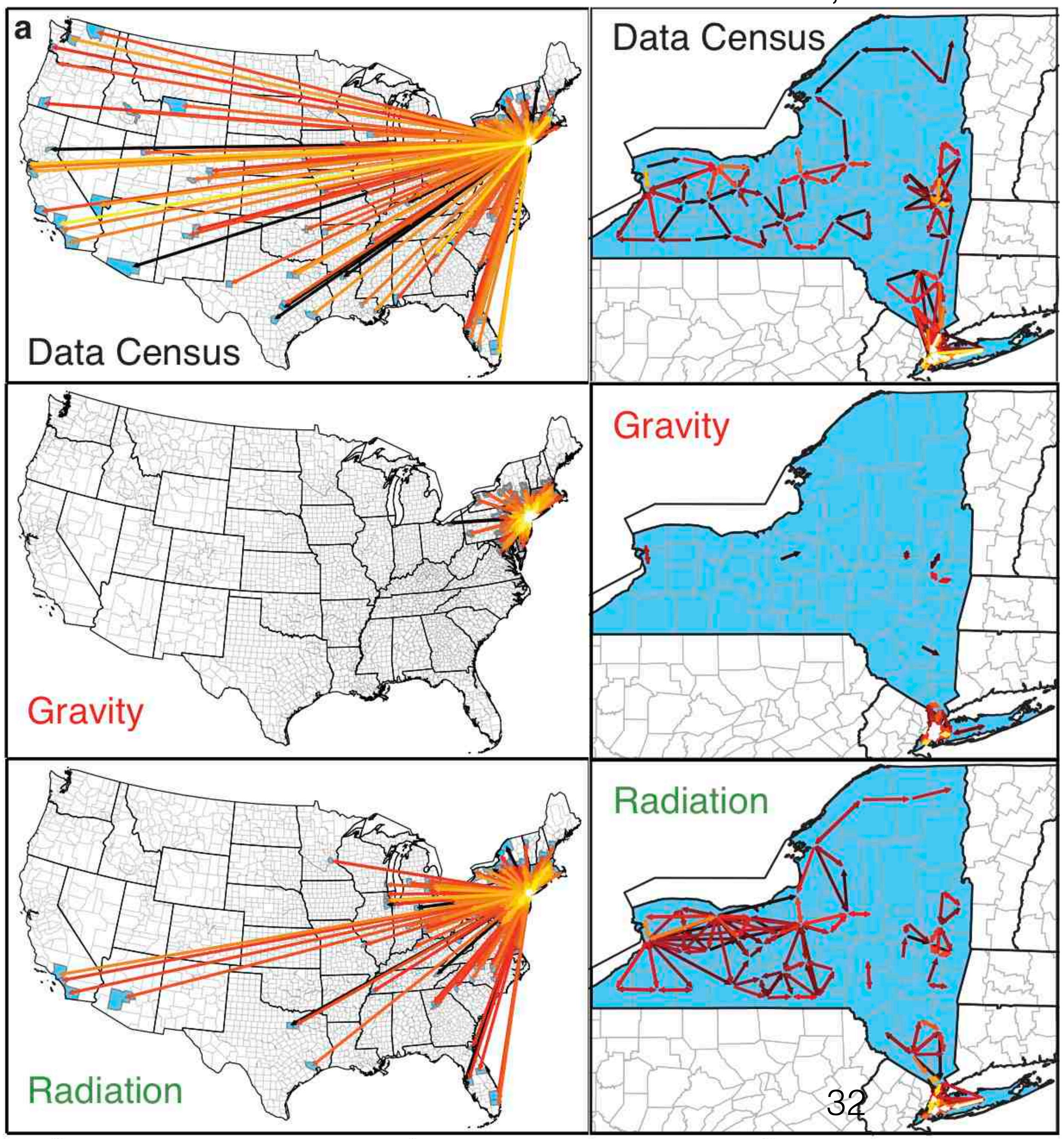
Illustration of the zone s_{ij} in which opportunities decrease the probability of interactions between i and j.

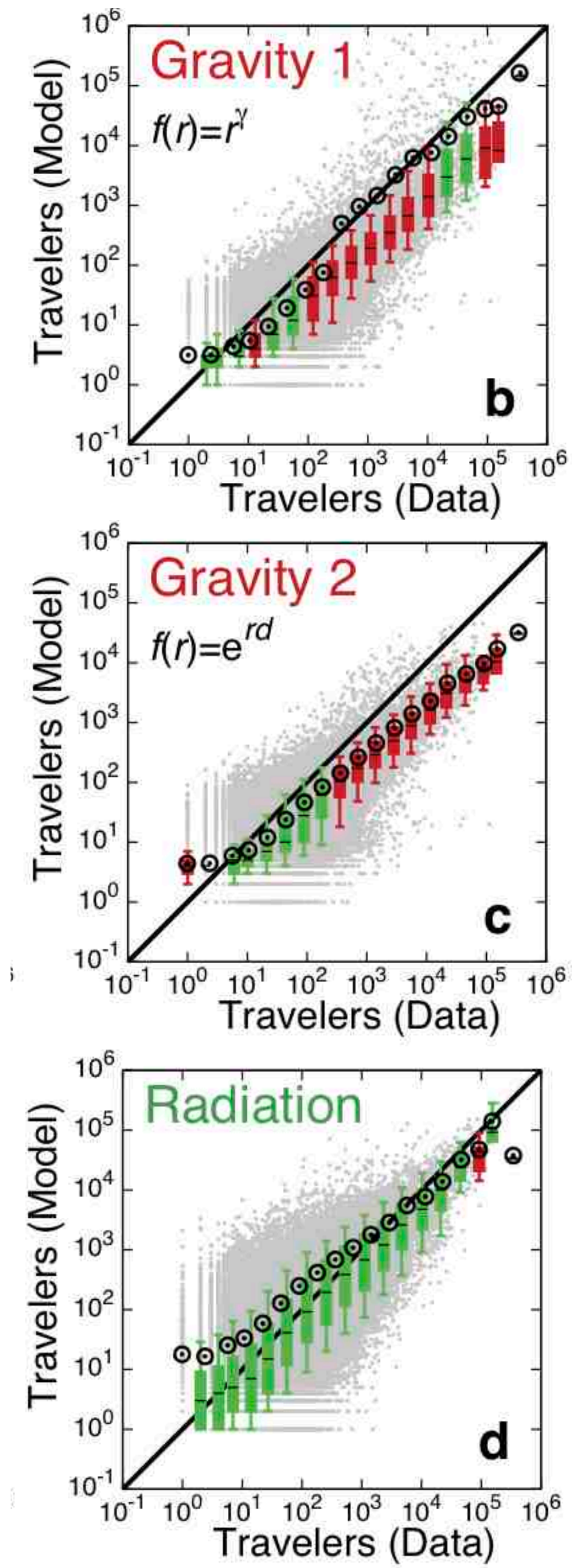


The radiation law

Comparison with census data and the gravity law predictions

Simini. et.al, Nature 2010





Radiation Law VS Gravity Law

+ Radiation:
  - No parameters
  - Two nodes of same degrees at similar distance can have different edge probability based on their location
+ Gravity:
  - Customizable deterrence function… The real world is complex !

MULTI-PARTITE GRAPHS

- Bi-partite: there exists 2 kinds of nodes, and links can only connect nodes of different types
  - Multi-partite: similar but with more than 2 types. (less common)
  - Not strictly different from normal graphs: if you don't know the two categories of nodes, it looks like any network
- Bi-partite networks are quite commonly use
  - Actors - Films
  - Clients - Products
  - Reserchers - conferences/institutions
  - …

MULTI-PARTITE GRAPHS

- The problem is that some definitions of normal graphs become meaningless
  - Clustering coefficient
  - Modularity
  - …

MULTI-PARTITE GRAPHS

Modularity: do not count pairs of nodes of same types

$$Q_B = \frac{1}{m}\sum_{u=1}^{r}\sum_{v=1}^{c}(\tilde{A}_{uv} - P_{uv})\delta(g_u, h_v) = \frac{1}{m}\sum_{u=1}^{r}\sum_{v=1}^{c}\left(\tilde{A}_{uv} - \frac{k_u d_v}{m}\right)\delta(g_u, h_v),$$

MULTI-PARTITE GRAPHS

Clustering Coefficient:

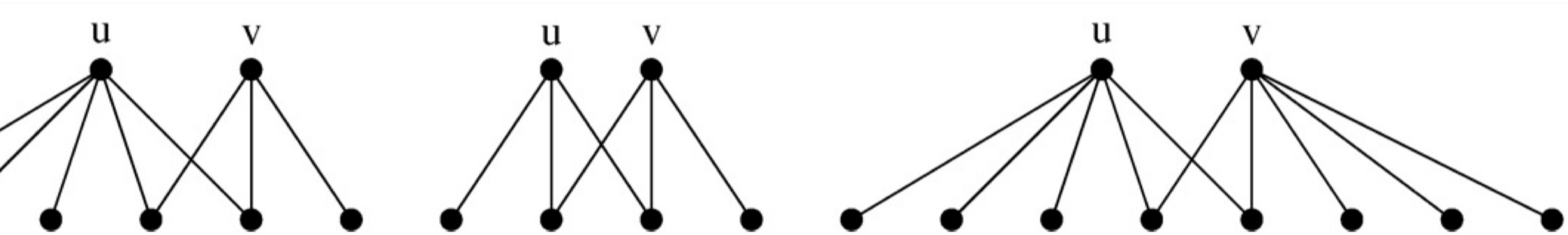
Of a pair

$$\mathrm{cc}_\bullet(u, v) = \frac{|N(u) \cap N(v)|}{|N(u) \cup N(v)|}$$

Of a Node: Average among nodes N at distance 2

$$\mathrm{cc}_\bullet(u) = \frac{\sum_{v \in N(N(u))} \mathrm{cc}_\bullet(u, v)}{|N(N(u))|}$$

cc(u,v)



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MULTI-PARTITE GRAPHS

- Large literature on the topic, in particular applications to **recommendation**
  - Users - products => propose the right products to the right user

Kunegis, J., De Luca, E. W., & Albayrak, S. (2010, June). The link prediction problem in bipartite networks. In *International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems* (pp. 380-389). Springer, Berlin, Heidelberg.

Barber, M. J. (2007). Modularity and community detection in bipartite networks. *Physical Review E*, *76*(6), 066102.

Zhang, P., Wang, J., Li, X., Li, M., Di, Z., & Fan, Y. (2008). Clustering coefficient and community structure of bipartite networks. *Physica A: Statistical Mechanics and its Applications*, *387*(27), 6869-6875.

MULTI-PARTITE GRAPHS

- A bipartite graph can be **projected** on one of its node-set
- One set of of nodes remain as nodes
- Those nodes are connected if they share a neighbor in the bipartite graphs
  - Variations: threshold, corrected by a null-model, etc.