## GRAPH NEURAL NETWORKS

Wu, Z., Pan, S., Chen, F., Long, G., Zhang, C., \& Yu, P. S. (2019). A comprehensive survey on graph neural networks. arXiv preprint arXiv:1901.00596.
Zhang, Z., Cui, P., \& Zhu, W. (2018). Deep learning on graphs: A survey. arXiv preprint arXiv:1812.04202.

Kipf, T. N., \& Welling, M. (2016). Semi-supervised classification with graph convolutional networks. arXiv preprint arXiv:1609.02907.

## WHY GRAPHS

- Graphs are everywhere!
- Molecules: atoms as nodes and bonds as edges
- Analyze molecular structure
- Knowledge Graphs
- Discover new facts or missing relationships between entities
- Recommender Systems
- Bipartite graphs, User/Item
- Traffic/Transport networks
- Predict traffic jams, impact of an exceptional event, etc.


## WHY GRAPHS

- Graph tasks
- Link prediction (follow recommendation, drug/illness relationship...)
- Node classification (bot detection in social media)
- Attribute regression (age of individuals in a social media...)
- Link classification/regression (relationship is: friend/colleague/lover ?)
- Graph classification (Molecule classification)
- Community detection


## WHY NN

- Neural networks are especially useful with structured data
- Images (each pixel has left/right/top/bottom pixels)
- Text (each word has a specific position in a sentence)
- Graphs are pure structure!


## GRAPH CONVOLUTION

- GCN : Graph Convolutional Network
- An adaptation of the Convolution used on images to graphs
- Kipf,T. N., \& Welling, M. (2016). Semi-supervised classification with graph convolutional networks. arXiv preprint arXiv: 1 609.02907.


## CONVOLUTION

| $1_{x y}$ | $1_{x 0}$ | $1_{x}$ | 0 | 0 |
| :--- | :--- | :--- | :--- | :--- |
| $0_{0}$ | $1_{x}$ | $1_{1}$ | 1 | 0 |
| $0_{x a}$ | $0_{0}$ | $1_{x}$ | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |

Image


Convolved
Feature

- Extract "features" of "higher level"
- Pixels => lines, curves, dots => circles, long lines, curvy shapes => eye, hand, leaves => Animal, Car, sky ...


## CONVOLUTION

- A convolution is defined by the weights of its kernel
- Which kernel(s) should we use?
- Weights of the kernel can be learnt, too

| Identity | $\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$ | , |
| :---: | :---: | :---: |
| Edge detection | $\left[\begin{array}{rrr}1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1\end{array}\right]$ |  |
|  | $\left[\begin{array}{rrr}0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0\end{array}\right]$ |  |
|  | $\left[\begin{array}{rrr}-1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1\end{array}\right]$ |  |
| Sharpen | $\left[\begin{array}{rrr}0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0\end{array}\right]$ |  |
| Box blur (normalized) | $\frac{1}{9}\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1\end{array}\right]$ |  |
| Gaussian blur $\mathbf{3 \times 3}$ (approximation) | $\frac{1}{16}\left[\begin{array}{lll}1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1\end{array}\right]$ |  |

https://en.wikipedia.org/wiki/Kernel_(image_processing)

## CONVOLUTIONAL NEURAL NETWORK



## cONVOLUTIONAL NEURAL

 NETWORK- Convolution on a picture can be seen as a special case of a graph operation:
- Combine weights of neighbors
- With an image represented as a regular grid


## DIFFERENCES

- In networks, number of neighbors different for each node
- Impossible to have a "fix" convolution kernel
- Matrix representations of images vs graphs
- Same object, completely different interpreation
- Graphs: position in the matrix (row, column) has no meaning
- Invariance to node ordering



## GRAPH CONVOLUTION


(a) 2D Convolution. Analogous to a graph, each pixel in an image is taken as a node where neighbors are determined by the filter size. The 2D convolution takes a weighted average of pixel values of the red node along with its neighbors. The neighbors of a node are ordered and have a fixed size.

(b) Graph Convolution. To get a hidden representation of the red node, one simple solution of graph convolution operation takes the average value of node features of the red node along with its neighbors. Different from image data, the neighbors of a node are unordered and variable in size.

Fig. 1: 2D Convolution vs. Graph Convolution.

## GRAPH CONVOLUTION

- Message passing interpretation
- Each node sends its information to its neighbors
- Nodes "combine" (convolution) their neighbors' information (+ their own) to construct new features
- Tell me who your friends are, l'll tell you who you are
- Can be related to:
- Information Diffusion on Networks
- PageRank
- Label propagation algorithms
- ...


## GCN LAYER INTUITION

- Convolution in images:
- I)Computes directly a weighted sum of neighbors' values
- Learn the proper weights
- 2)Often followed by pooling
- Convolution in graphs:
- Weights cannot be learned directly
- I)Average the neighbors' features (pooling-like)
- Using fix, predefined weights
- 2)Computes the weighted sum of neighbors' values
- Learn the proper weights


## GCN LAYER INTUITION

- A graph convolution can be understood as a linear (fully connected) layer, with:
- As input the average features of the neighbors
- As output an embedding in the desired number of dimensions
- Equivalent to the number of neurons in a linear layer
- But also intepratable as the number of channels in Conv layer


## GRAPHS $=$ INDEPENDENT ITEMS DATASET

- Graphs are inherently different from image/tabular datasets
- Images/tabular
- Each item is independent of the others
- => We train for each item independently
- => The test set is composed of new, never-seen items
- Graphs (general case)
- A single graph, composed of (connected) nodes
- =>Each node is treated as an independent item
- =>But all nodes features are used in training
- =>Only target can be split in training/test
- =>"Semi-supervised learning"


## GRAPHS $=$ INDEPENDENT ITEMS DATASET

- Example: Network ofTwitter users
- Nodes: users
- Edges: followers
- Attributes: date joined, likes, geographical position, keywords,...
- Target: Male/Female, Left/Right, etc.
- We know it for some users, but not all
- Using all users' properties to guess the target for some users, training on the known one


## GRAPH CONVOLUTION

## Stacking convolution layers



## GRAPH CONVOLUTION

- Each convolution layer allows to depend on nodes farther in the network
- Layer I: results depend only on direct neighbors
- Layer 2:
- direct neighbors' features are result of Layer I
- =>results depends on nodes at distance I and 2
- Etc.
- Similar as convolutions in images



## GRAPH CONVOLUTION

- Good news: average distance in real graphs is short - 6 degrees of separation
- Even on a large graph, a moderate number of convolutional layers should allow to have impact from most of the graph

GCN EQUATION

## GRAPH CONVOLUTION

$H^{(l+1)}=f\left(H^{(l)}, A\right)$
$f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right)$
$H$ :node features
A: adjacency matrix $(\hat{A}=A+I)$
$l$ : layer index
$D$ : Degree matrix (degrees on the diagonal)
$W$ : learnable weights
$\sigma$ : activation fonction (often ReLU)

## ADJACENCY MATRIXA



## NORMALIZED A



$$
D^{-1} \hat{A}
$$

Performs an average


$$
D^{-\frac{1}{2}} \hat{A} D^{-\frac{1}{2}}
$$

Average weighted by degree
Normalisation of the adjacency matrix

$$
\begin{gathered}
f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right) \\
H W=C O M B I N E F E A T U R E S \\
\hline
\end{gathered}
$$

$$
f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right)
$$

## $A(H W)=A V E R A G E$ OVER NEIGHBORS

|  | $e^{\text {e }}$ |  |
| :---: | :---: | :---: |
| n1 | w |  |
| n2 | X |  |
| n3 | y |  |
| n4 | z |  |



$$
f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right)
$$

## A(HW) = AVERAGE NEIGHBORS EMBEDDING

(AH)W= EMBED AVERAGE OF NEIGHBORS FEATURES

MATRIX MULTIPLICATION IS ASSOCIATIVE

## GRAPH CONVOLUTION

- Individual embeddings computed as
$h_{i}^{l+1}=\sum_{j \in N_{i}} \frac{1}{\sqrt{\operatorname{deg}(i)} \sqrt{\operatorname{deg}(j)}} h_{j}^{l} W^{T}$
- $h_{j}^{l}$ embedding of node $j$ in the previous layer
- Embedding of node $i$ is a weighted sum of its neighbors' attributes multiplied by weights


## ACTIVATION FUNCTION $f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right)$

$\sigma$ is an activation function.
It is used to introduce non-linearity.
A common choice is to use the ReLU, (Rectified Linear Unit)
$=>$ Simple to differentiate and to compute


# ACTIVATION FUNCTION $f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right)$ 

$\sigma$ is an activation function.
It is used to introduce non-linearity.
Remember, Multiplication is associative

```
((X*W1 + b1)*W2 + b2)*W3 + b3
(X*W1*W2+b1*W2 + b2)*W3 + b3
(X*W1*W2*W3 + b1*W2*W3 + b2*W3) + b3
(X*W1*W2*W3) + (b1*W2*W3 + b2*W3 + b3)
```

( $\mathrm{X}^{*} \mathrm{~W}$ ) +B
Where, $\mathrm{W}=\mathrm{W} 1^{*} \mathrm{~W} 2^{*} \mathrm{~W} 3$ and $\mathrm{B}=\mathrm{b} 1^{*} \mathrm{~W} 2^{*} \mathrm{~W} 3+\mathrm{b} 2^{*} \mathrm{~W} 3+\mathrm{b} 3$

## GCN: STEP-BY-STEP

Without features: Structure only

## LAYERS SIZE <br> $$
f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right)
$$

Size of the weight matrices by layer

$$
\begin{aligned}
& W_{0}: d_{0} \times d_{1} \\
& W_{1}: d_{1} \times d_{2}
\end{aligned}
$$

$$
W_{n}: d_{n} \times d_{n+1}
$$

$d_{0}$ is the number of features per node in the original network data, $d_{n+1}$ is the number of desired features (usually followed by a normal classifier, e.g., logistic)

## FORWARD STEP

- We can first look at what happens without weight learning, i.e., doing only the forward step.
- We set the original features to the identity matrix, $H_{0}=I$. Each node's features is a one hot vector of itself ( 1 at its position, 0 otherwise)
- Weights are random (normal distribution centered on 0)
- Two layers, with $W$ sizes $n \times 5,5 \times 2$

$$
\text { FORWARD STEPA } \underset{f\left(H^{(l)}, A\right)=\sigma\left(\hat{D}^{-\frac{1}{2}} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)}\right)}{ }
$$

$\mathrm{LI}=\mathrm{n}$ to 5 features

$\mathrm{LI}=5$ to 2 features


## FORWARD STEP



Even with random weights, some structure is preserved in the "embedding" (colors=communities)

## FORWARD STEP



Why is some information preserved?
=>Label propagation mechanism, due to local structure (communities, transitivity...), close nodes receive similar values, convergence to a particular value...

## FORWARD STEP

K-means on the 2D "embedding"
(paramater $\mathrm{k}=3$ clusters)


Node positions based on spring layout, colors=clusters

## BACKWARD STEP

- To learn the weights, we use back-propagation
- Short summary
- A loss function is defined to compare the "predicted values" with ground truth labels (at this point, we need some labels...)
- The derivative of the cost function relative to weights is computed
- Weights are updated using grading descent (i.e., weights are modified in the direction that will minimize the loss)


## FITTING THE GCN

- We define a "semi-supervised" objective:
- Labels are known only for a few nodes (the 2 instructors)
- Choose a loss function for binary classification (logistic...)
- The loss is computed only for the two instructors
- We run e steps ("epoch'") of back-propagation, until convergence


## FITTING THE GCN



Step I:
Combine one-hot to 5D

Step2:
Combine 5D to 2D


Result:
Computed feature vector
As expected, values for nodes
0 and 33 are opposed

## FITTING THE GCN

Epoch 0 | Loss: 0.6987 Epoch 1 | Loss: 0.6804 Epoch 2 | Loss: 0.6634 Epoch 3 | Loss: 0.6476 Epoch 4 Loss: 0.632
Epoch 4 Loss: 0.632 Epoch 5 Loss: 0.6174

Epoch 6 Loss: 0.6017 | Epoch 6 | Loss: 0.6017 |
| :--- | :--- |
| Epoch 7 | Loss: 0.5852 | Epoch 7 Loss: 0.5852 Epoch 8 Loss: 0.5684 Epoch 9 Loss: 0.5513 Epoch 10 | Epoch 11 I Loss: 0.5338 Loss: 0.5158 Loss: 0.4976 Loss: 0.4792 Loss: 0.4605 Loss: 0.4416 Loss: 0.4225 Loss: 0.4033 Loss: 0.3842 Loss: 0.3652 Loss: 0.3464 Loss: 0.3464

Loss: 0.3279 Loss: 0.3279
Loss: 0.3096 Loss: 0.3096 Loss: 0.2916 Loss: 0.2571 Loss: 0.2407 Loss: 0.2248 Loss: 0.2095 Loss: 0.1946 Loss: 0.1803 Loss: 0.1668 Loss: 0.1541 Loss: 0.1422 Loss: 0.1312 Loss: 0.1312 Loss: 0.1209 Loss: 0.1113 Loss: 0.1024 Loss: 0.0940
Loss: 0.0863 Loss: 0.0793 Loss: 0.0727 Loss: 0.0667 Loss: 0.0611 Loss: 0.0560 Loss: 0.0513 Loss: 0.0470 Loss: 0.0432 Loss: 0.0396 Loss: 0.0363 Loss: 0.0333

Epoch: 0


## RESULTS

Features values


## Highest feature as <br> label



We retrieve the expected
"communities"

## GAT

Graph ATtention networks

## SELF-ATTENTION MECHANISM

- Mechanisms coming mostly from Language models
- Transformers (as in GPT) are a particular type of self-attention


## GRAPH ATTENTION

- In the normal GCN, a limit is the fix rule used to combine the neighbors attributes (weighted average)
, $h_{i}^{l+1}=\sum_{j \in N_{i}} \frac{1}{\sqrt{\operatorname{deg}(i)} \sqrt{\operatorname{deg}(j)}} h_{j}^{l} W^{T}$
- Graph attention principle is to allow each node to "choose" what "attention" to give to each neighbor
$h_{i}^{l+1}=\sum_{j \in N_{i}} \alpha_{i j} h_{j}^{l} W^{T}$
- $\alpha_{i j}$ attention from $i$ to $j$


## GRAPH ATTENTION

- Step I: a learnable attention matrix convert the node embeddings into new embeddings specific for Attention
- $z_{i}=W h_{i}$
- $W$ the learnable weights
- $h_{i}$ existing nodes features (embeddings/attributes)
- $=>$ We don't want the node embedding to combine the meanings of:
- The node position in the graph
- The way it behaves in term of attention to others


## GRAPH ATTENTION

- Step 2: concatenate both nodes embedding
- $z_{i} \| z_{j}$
- [a,b,c] || [f,e,d] => [a,b,c,f,e,d]
- => To decide if a link is important, we will consider the attributes/embeddings of both nodes


## GRAPH ATTENTION

- Step 2: Compute an attention coefficient $a_{i j}$ using learnable weights
- $e_{i j}=a^{T}\left[z_{i}| | z_{j}\right]$
- $a$ is a learnable vector, common to all node pairs, computing the attention from the combined embedding
- => Same principle as a perceptron/linear regression
- => Variant (as in transformer): compute $z_{i} \cdot z_{j}$
- Step 3: Add an activation function
- $e_{i j}=\operatorname{ReLu}\left(a^{T}\left[z_{i} \| z_{j}\right]\right)$


## GRAPH ATTENTION

- Step 4: Softmax normalization
- We have unnormalized attention score for each neighbor
- Use Softmax to normalize attention
- small value for a single node $n, 0$ value for all others $=>$ all attention on $n$ !
$\alpha_{i j}=\operatorname{softmax}\left(e_{i j}\right)=\frac{\exp \left(e_{i j}\right)}{\sum_{k \in N_{i}} \exp \left(e_{i k}\right)}$


## GRAPH ATTENTION

- Finally: Multi-head attention
- A single attention layer might not be powerful enough. What we described is called an attention head, and we typically have multiple heads
- The resulting embeddings are then combined

$$
\text { Average (i.e., } h_{i}=\frac{1}{n} \sum_{k=1}^{n} h_{i}^{k} \text { ) }
$$

- Similar principle as the multiple channels of a convolution



## GRAPH AUTOENCODERS

## AUTOENCODERS

- Autoencoders are mostly used for unsupervised learning using deep neural networks
- Typically, for images.
- Composed of two parts
- An encoder
- e.g., a classic sequence of convolutional layers
- A decoder
- e.g., an inverse architecture (e.g., the same layers in inverse order)
- In the middle is the "embedding", what we are interested in
- Constrained to be small


## AUTOENCODERS



## AUTOENCODERS

- The objective is to
- Encode a complex object
- e.g., a 3 color layers, $256 \times 256$ image
- Into a small-dimensional vector
- e.g., vector of size 128
- Such that these vectors allow to reproduce the output with minimal loss of information
- Many applications:
- Visualization (like PCA)
- Downstream task (these vectors can be used for classification, etc.)
- Generate variations (Generative image models...)


## GRAPH AUTOENCODERS

- Same principle, but with graphs :)
- Classic architecture[I]:
- Encoder: GCN layers (e.g., 2 layers)
- Decoder: Dot product between embeddings (+activation)
- Minimize the binary cross entropy between input and output adjacency matrices
- =>Compute vectors for each node
- such that their dot product is
- Close to I if they are connected (parallel => similar vectors)
- Close to 0 if they are not (orthogonal => different vectors)


## VARIANT:VGAE

- VAE :Variational AutoEncoder
- Popular improvement over classic AutoEncoder
- Limits of Autoencoders:
- Embedding space is often poorly structured
- Poor continuity:The "middle" vector between two vectors ( $\mathrm{V} 1, \mathrm{~V} 2$ ) do not correspond to a middle image between the two corresponding to $\mathrm{vI} / \mathrm{v} 2$
- Poor completeness: Space seems "sparse": many vectors correspond to nothing meaningful
- VAE solution:
- Instead of encoding an input as a single point, we encode it as a distribution over the latent space


## VAE



## VAE



## VAE

- The model is trained as follows:
- I)the input is encoded as gaussian distribution over the latent space
- 2) a point from the latent space is sampled from that distribution
- 3) the sampled point is decoded and the reconstruction error can be computed
- 4) finally, the reconstruction error is backpropagated through the network


## VAE

Regularization: trade-off between best fit to data and distance between each gaussian and a standard gaussian(centered, unit variance)

what can happen without regularisation
what we want to obtain with regularisation


## VGAE

- Simple adaptation to graphs, i.e., a classic graph autoencoder in which the encoding part is replaced by Variational mechanism.
- In practice:
- Layer I: normal GCN
- Layer 2: two parallel GCN layers
- One to learn the centroid
- One to learn the variance (diagonal of the covariance matrix)
- =>For each node, instead of having I vector of size d, we have two vectors of size d
- To decode, we take a random point from the multivariate gaussian


## LINK PREDICTION

## LINK PREDICTION

- Observed network: current state
- Link prediction:What edge
- Might appear in the future (future link prediction)
- Might have been missed (missing link prediction)
- Many applications
- Recommender systems
- Drug/healness prediction, ...


## LINK PREDICTION

- Classification objective
- Binary classes: edge/No edge
- Usually, evaluation based on class probability
- AUC,AP...
- Evaluation process
- Hide some of the edges in the graph
- Check that
- Training on the remaining edges
- We predict well the removed ones


## LINK PREDICTION

- Classic methods
- Common Neighbors
- Adamic Adar
- ...
- =>Work only on nodes at distance two
- Advanced methods
- Graph embedding (DeepWalk, Node2Vec)
- Use dot product of embedding as score, or other variants, e.g., training a classification on vectors
- Community structure, random walks
- =>Do not take node features into account


## LINK PREDICTION

- Using VGAE
- The objetive ofVGAE is to reconstruct the graph, i.e., to predict which edge is present or not =>Directly a link prediction objective
- VGAE final step: dot product of embeddings
- Edge prediction score: result of the dot product of node vectors


## LINK PREDICTION

- Using directly a GNN
- GNNs produce node embeddings in the output
- We need to combine node embeddings
- Two (main) solutions
- Create a combined vector from two independent vectors, and add a linear layer for classification
- Use directly a vector-to-scalar operation


## LINK PREDICTION

- Combining two node vectors into a node-pair vector
- Vector concatenation $[\times 1, \times 2][\times 3, \times 4]=>[\times 1, \times 2, \times 3, \times 4]$
- LI difference $[x 1, \times 2][\times 3, \times 4]=>[x 1-\times 3, \times 2-\times 4]$
- Hadamard Product $[x 1, \times 2][x 3, \times 4]=>[\times 1 * \times 3, \times 2 * \times 4]$
- Followed by a classification task on this new vector


## LINK PREDICTION

- Combining two node vectors into a scalar
dot product $\approx$ unnormalized cosine similarity



## TRANSDUCTIVE / INDUCTIVE

- Transductive
- What we discussed until now:
- We have access to the whole graph at training time
- We just don't see all the labels (test, prediction)
- Inductive
- Train on a set of nodes/graphs
- Results can be applied to unseen nodes/graphs
- A GCN layer can be trained on multiple (sub)networks, and learned weights used on a new scenario (but not very efficient)
- GraphSAGE=>Works for each node on a local graph centered on the node, by sampling a fixed number of neighbors. Transform the graph problem in a more classic problem.


## MULTI-PARTITE GRAPHS

- Nodes of multiple types:
- Items/Users
- Drug/illness
- Each type of node has their own attributes
- Cannot learn a single GCN layer
- =>Learn 2 independent layers
- User attributes to Item attributes
- Item attributes to User attributes

