GRAPH NEURAL NETWORKS

Wu, Z., Pan, S., Chen, F., Long, G., Zhang, C., & Yu, P. S. (2019). A comprehensive survey on graph neural networks. *arXiv preprint arXiv:1901.00596*.
 Zhang, Z., Cui, P., & Zhu, W. (2018). Deep learning on graphs: A survey. *arXiv preprint arXiv:1812.04202*.

Kipf, T. N., & Welling, M. (2016). Semi-supervised classification with graph convolutional networks. arXiv preprint arXiv:1609.02907.

WHY GRAPHS

• Graphs are everywhere!

- Molecules: atoms as nodes and bonds as edges
 - Analyze molecular structure
- Knowledge Graphs
 - Discover new facts or missing relationships between entities
- Recommender Systems
 - Bipartite graphs, User/Item
- Traffic/Transport networks
 - Predict traffic jams, impact of an exceptional event, etc.

WHY GRAPHS

Graph tasks

. . .

- Link prediction (follow recommendation, drug/illness relationship...)
- Node classification (bot detection in social media)
- Attribute regression (age of individuals in a social media...)
- Link classification/regression (relationship is: friend/colleague/lover ?)
- Graph classification (Molecule classification)
- Community detection

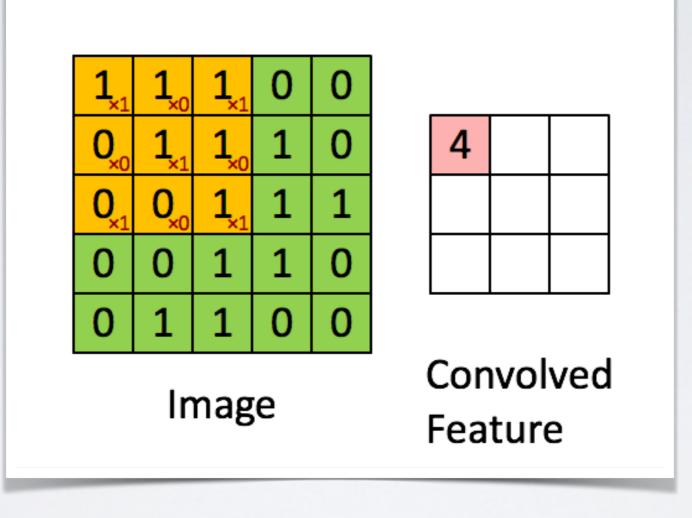
WHY NN

- Neural networks are especially useful with structured data
 - Images (each pixel has left/right/top/bottom pixels)
 - Text (each word has a specific position in a sentence)
- Graphs are pure structure!

GCN : Graph Convolutional Network

- An adaptation of the Convolution used on images to graphs
- Kipf, T. N., & Welling, M. (2016). Semi-supervised classification with graph convolutional networks. arXiv preprint arXiv:1609.02907.

CONVOLUTION



- Extract 'features' of 'higher level'
 - Pixels => lines, curves, dots => circles, long lines, curvy shapes => eye, hand, leaves => Animal, Car, sky ...

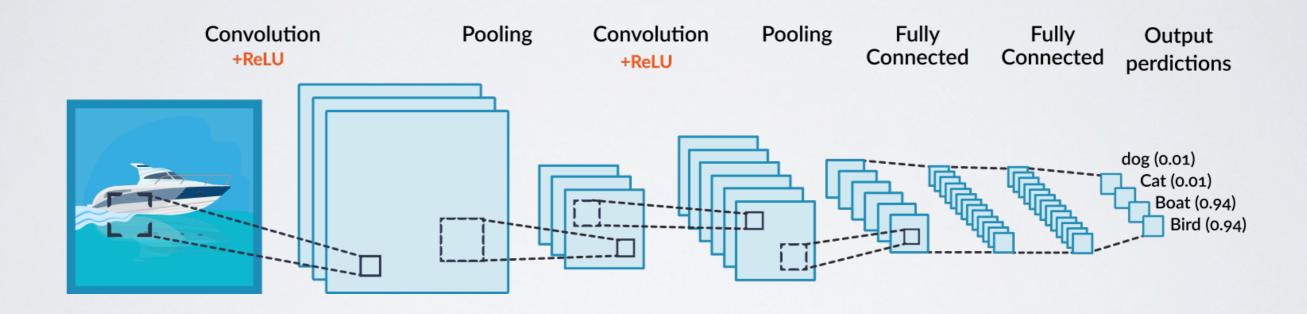
CONVOLUTION

- A convolution is defined by the weights of its kernel
- Which kernel(s) should we use?
- Weights of the kernel can be learnt, too

Identity	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
Edge detection	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$	
	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	
	$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	
Sharpen	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	
Box blur (normalized)	$\frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	Contraction of the second
Gaussian blur 3 × 3 (approximation)	$\frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	

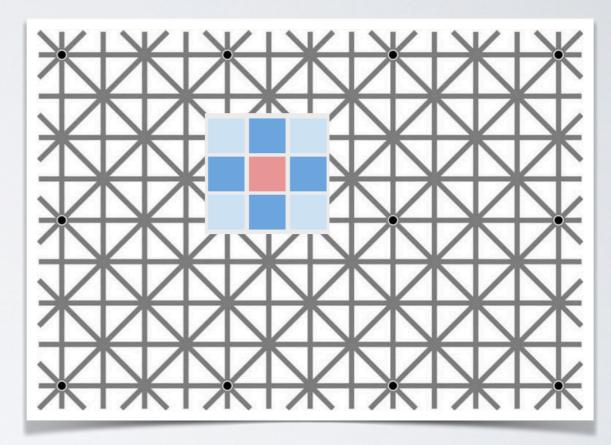
https://en.wikipedia.org/wiki/Kernel_(image_processing)

CONVOLUTIONAL NEURAL NETWORK



CONVOLUTIONAL NEURAL NETWORK

- Convolution on a picture can be seen as a special case of a graph operation:
 - Combine weights of neighbors
 - With an image represented as a regular grid

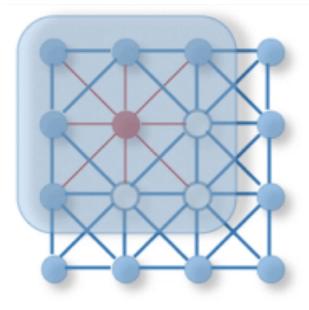


https://www.inference.vc/how-powerful-are-graph-convolutions-review-of-kipf-welling-2016-2/

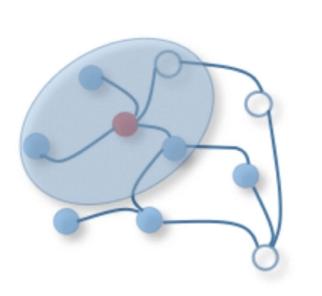
DIFFERENCES

- In networks, number of neighbors different for each node
 Impossible to have a 'fix' correction of the second second
- Matrix representations of images vs graphs
 - reation
 - Same object, completely sliffe
 Graphs: position in the matrix
 Invariance to noce order. umn) has no meaning





(a) 2D Convolution. Analogous to a graph, each pixel in an image is taken as a node where neighbors are determined by the filter size. The 2D convolution takes a weighted average of pixel values of the red node along with its neighbors. The neighbors of a node are ordered and have a fixed size.



(b) Graph Convolution. To get a hidden representation of the red node, one simple solution of graph convolution operation takes the average value of node features of the red node along with its neighbors. Different from image data, the neighbors of a node are unordered and variable in size.

Fig. 1: 2D Convolution vs. Graph Convolution.

Message passing interpretation

- Each node sends its information to its neighbors
- Nodes "combine" (convolution) their neighbors' information (+ their own) to construct new features
- Tell me who your friends are, I'll tell you who you are
- Can be related to:
 - Information Diffusion on Networks
 - PageRank
 - Label propagation algorithms

• • • •

GCN LAYER INTUITION

Convolution in images:

- I)Computes directly a weighted sum of neighbors' values
 - Learn the proper weights
- 2)Often followed by pooling
- Convolution in graphs:
 - Weights cannot be learned directly
 - I)Average the neighbors' features (pooling-like)
 - Using fix, predefined weights
 - 2)Computes the weighted sum of neighbors' values
 - Learn the proper weights

GCN LAYER INTUITION

- A graph convolution can be understood as a linear (fully connected) layer, with:
 - As input the average features of the neighbors
 - As output an embedding in the desired number of dimensions
 - Equivalent to the number of neurons in a linear layer
 - But also intepratable as the number of channels in Conv layer

GRAPHS ≠ INDEPENDENT ITEMS DATASET

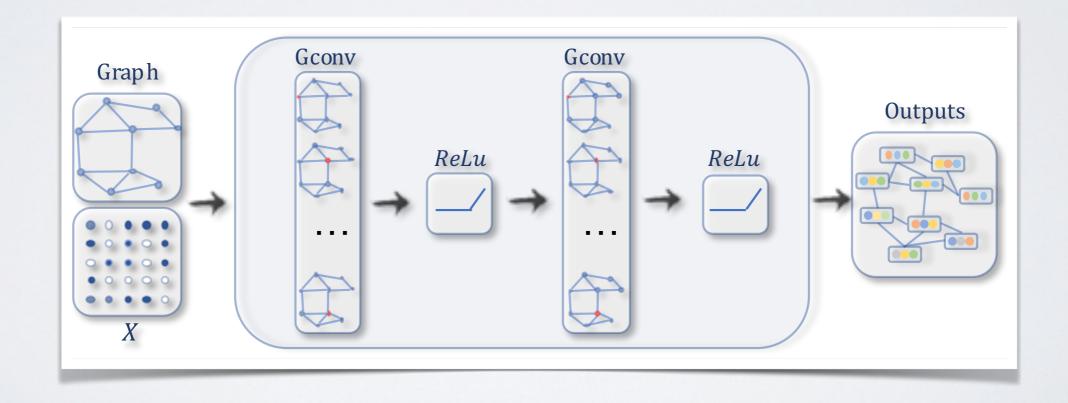
• Graphs are inherently different from image/tabular datasets

- Images/tabular
 - Each item is independent of the others
 - => We train for each item independently
 - => The test set is composed of new, never-seen items
- Graphs (general case)
 - A single graph, composed of (connected) nodes
 - =>Each node is treated as an independent item
 - =>But all nodes features are used in training
 - =>Only target can be split in training/test
 - =>"Semi-supervised learning"

GRAPHS ≠ INDEPENDENT ITEMS DATASET

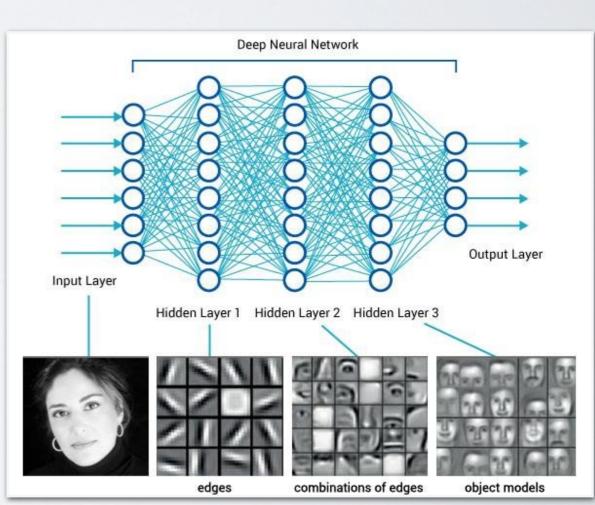
- Example: Network of Twitter users
 - Nodes: users
 - Edges: followers
- Attributes: date joined, likes, geographical position, keywords,...
- Target: Male/Female, Left/Right, etc.
 - We know it for some users, but not all
- Using all users' properties to guess the target for some users, training on the known one

Stacking convolution layers



Wu, Z., Pan, S., Chen, F., Long, G., Zhang, C., & Yu, P. S. (2019). A comprehensive survey on graph neural networks. arXiv preprint arXiv:1901.00596.

- Each convolution layer allows to depend on nodes farther in the network
 - Layer I: results depend only on direct neighbors
 - Layer 2:
 - direct neighbors' features are result of Layer I
 - =>results depends on nodes at distance I and 2
 - Etc.
- Similar as convolutions in images



- Good news: average distance in real graphs is short
 - 6 degrees of separation
- Even on a large graph, a moderate number of convolutional layers should allow to have impact from most of the graph

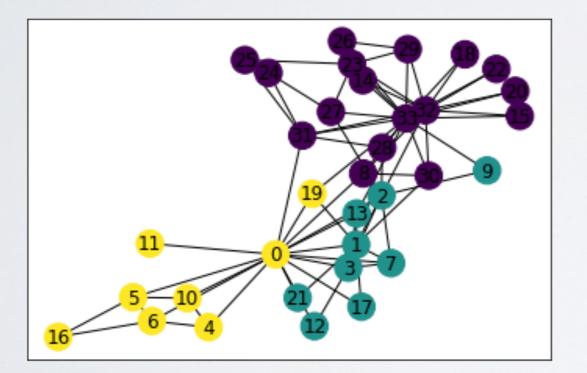
GCN EQUATION

GRAPH CONVOLUTION $H^{(l+1)} = f(H^{(l)}, A)$

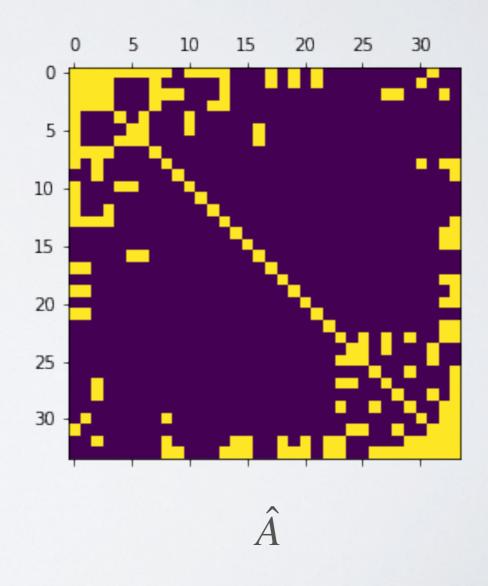
$$f(H^{(l)}, A) = \sigma\left(\hat{D}^{-\frac{1}{2}}\hat{A}\hat{D}^{-\frac{1}{2}}H^{(l)}W^{(l)}\right)$$

H: node features *A*: adjacency matrix $(\hat{A} = A + I)$ *l*: layer index *D*: Degree matrix (degrees on the diagonal) *W*: learnable weights σ : activation fonction (often ReLU)

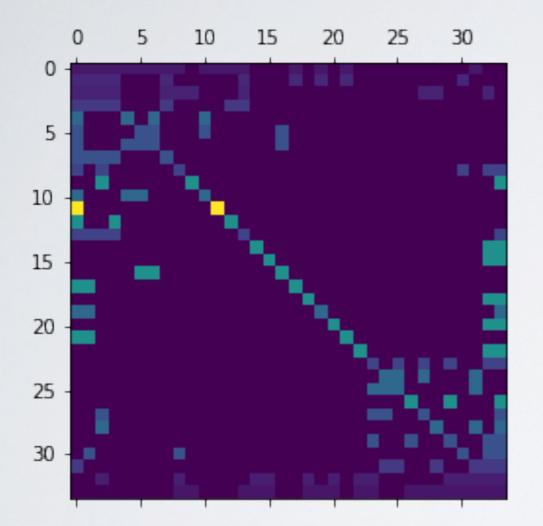
ADJACENCY MATRIX A

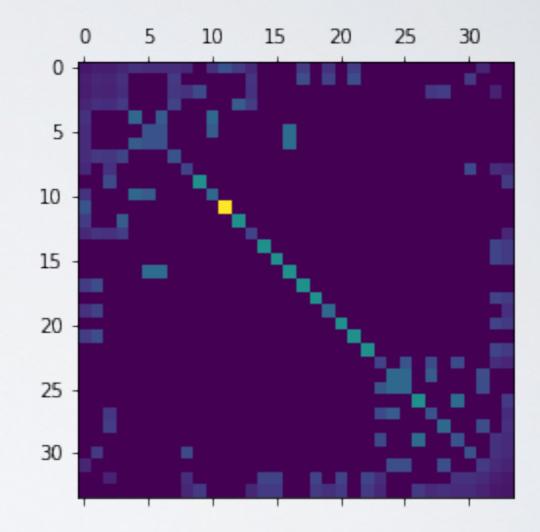


Zackary Karate club (with communities for reference)



NORMALIZEDA





$$D^{-1}\hat{A}$$

Performs an average

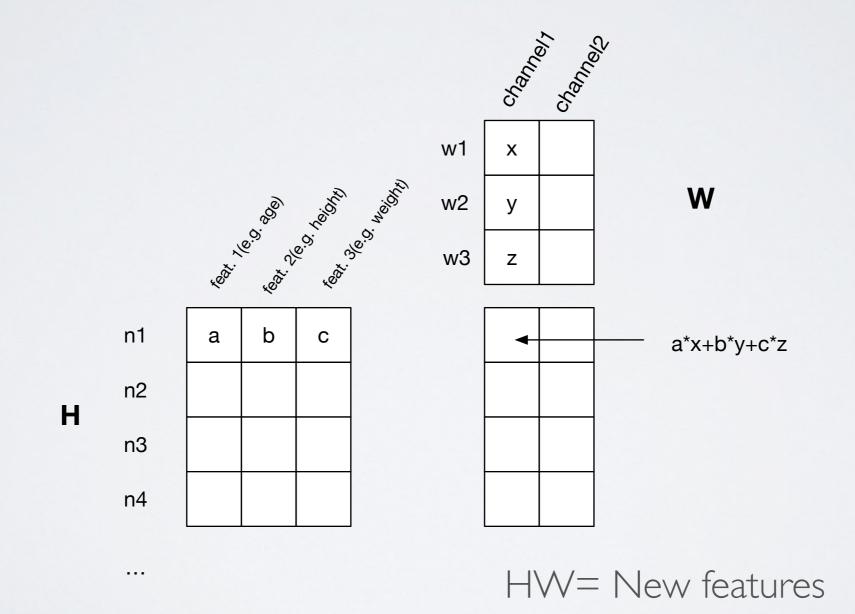
 $D^{-\frac{1}{2}}\hat{A}D^{-\frac{1}{2}}$

Average weighted by degree

Normalisation of the adjacency matrix

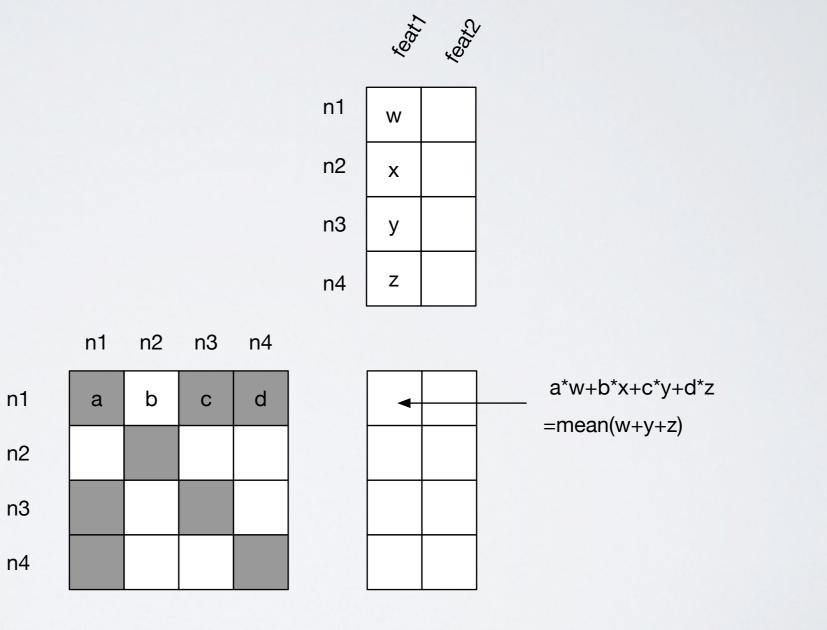
 $f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$

HW=COMBINE FEATURES



 $f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$

A(HW) = AVERAGE OVER NEIGHBORS





 $\hat{D}^{-\frac{1}{2}}\hat{A}$

1

 $f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$

A(HW) = AVERAGE NEIGHBORSEMBEDDING

(AH)W= EMBED AVERAGE OF NEIGHBORS FEATURES

MATRIX MULTIPLICATION IS ASSOCIATIVE

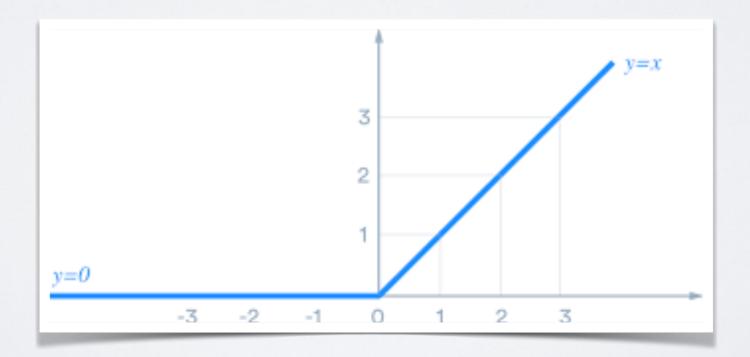
Individual embeddings computed as

$$h_i^{l+1} = \sum_{j \in N_i} \frac{1}{\sqrt{deg(i)}\sqrt{deg(j)}} h_j^l W^T$$

- h_j^l embedding of node j in the previous layer
- Embedding of node *i* is a weighted sum of its neighbors' attributes multiplied by weights

ACTIVATION FUNCTION $f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$

σ is an activation function.
 It is used to introduce non-linearity.
 A common choice is to use the **ReLU**, (Rectified Linear Unit)
 =>Simple to differentiate and to compute



https://medium.com/@danqing/a-practical-guide-to-relu-b83ca804f1f7

ACTIVATION FUNCTION $f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$

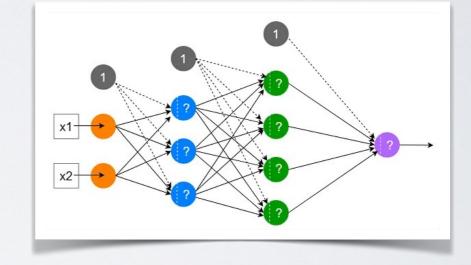
 σ is an activation function. It is used to introduce non-linearity.

Remember, Multiplication is associative

((X*W1 + b1)*W2 + b2)*W3 + b3

(X*W1*W2+ b1*W2 + b2)*W3 + b3 (X*W1*W2*W3 + b1*W2*W3 + b2*W3) + b3 (X*W1*W2*W3) + (b1*W2*W3 + b2*W3 + b3)

(X*W) + B Where, W = W1*W2*W3 and B = b1*W2*W3 + b2*W3 + b3



GCN: STEP-BY-STEP

Without features: Structure only

LAYERS SIZE
$$f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$$

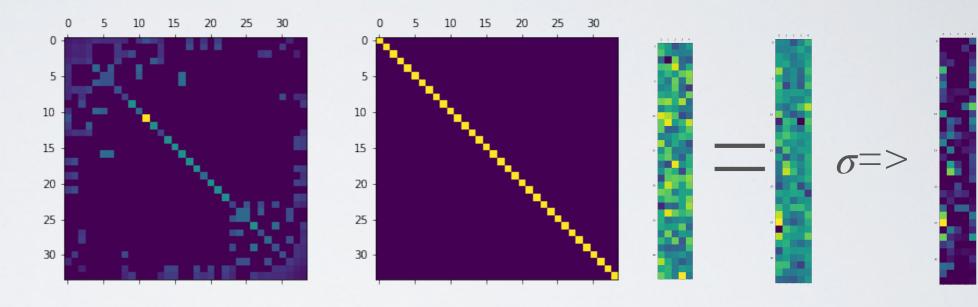
Size of the weight matrices by layer

$$W_0: d_0 \times d_1$$
$$W_1: d_1 \times d_2$$
$$\cdots$$
$$W_n: d_n \times d_{n+1}$$

 d_0 is the number of features per node in the original network data, d_{n+1} is the number of desired features (usually followed by a normal classifier, e.g., logistic)

- We can first look at what happens without weight learning, i.e., doing only the forward step.
- We set the original features to the identity matrix, $H_0 = I$. Each node's features is a *one hot vector* of itself (1 at its position, 0 otherwise)
- Weights are random (normal distribution centered on 0)
- Two layers, with W sizes $n \times 5, 5 \times 2$

FORWARD STEP $f(H^{(l)}, A) = \sigma \left(\hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} H^{(l)} W^{(l)} \right)$



LI = n to 5 features

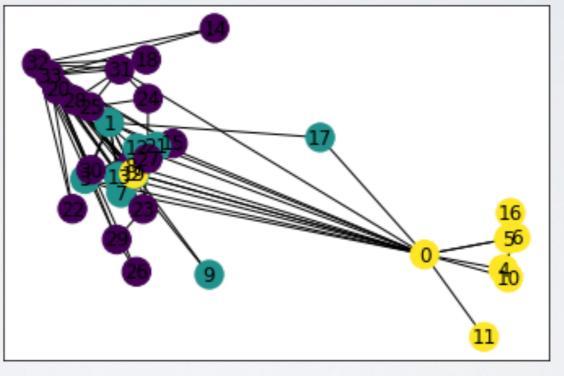
 0
 5
 10
 15
 20
 25
 30

 0
 -<



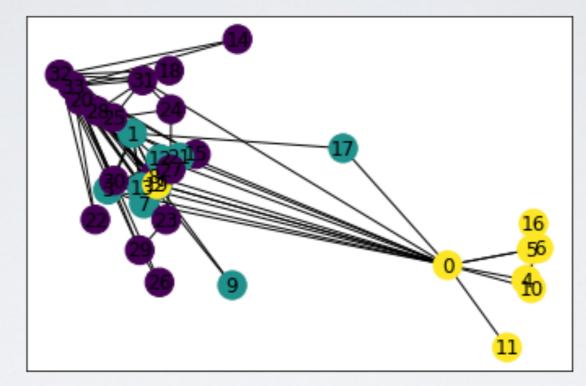
LI = 5 to 2 features





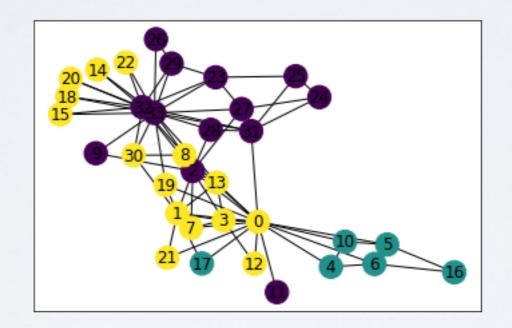
Dimension I

Even with random weights, some structure is preserved in the "embedding" (colors=communities)



Why is some information preserved? =>Label propagation mechanism, due to local structure (communities, transitivity...), close nodes receive similar values, convergence to a particular value...

K-means on the 2D "embedding" (paramater k=3 clusters)



Node positions based on spring layout, colors=clusters

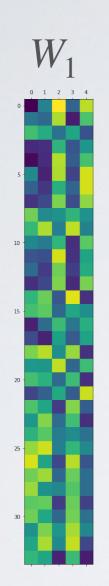
BACKWARD STEP

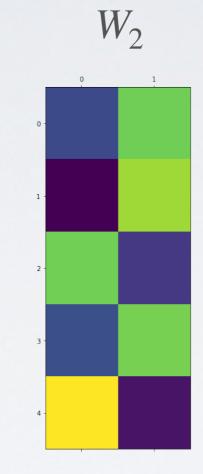
- To learn the weights, we use **back-propagation**
- Short summary
 - A **loss** function is defined to compare the "predicted values" with ground truth labels (at this point, we need some labels...)
 - The **derivative** of the cost function relative to weights is computed
 - Weights are updated using grading descent (i.e., weights are modified in the direction that will minimize the loss)

FITTING THE GCN

- We define a "semi-supervised" objective:
 - Labels are known only for a few nodes (the 2 instructors)
 - Choose a loss function for binary classification (logistic...)
 - The loss is computed only for the two instructors
- We run e steps ("epoch") of back-propagation, until convergence

FITTING THE GCN







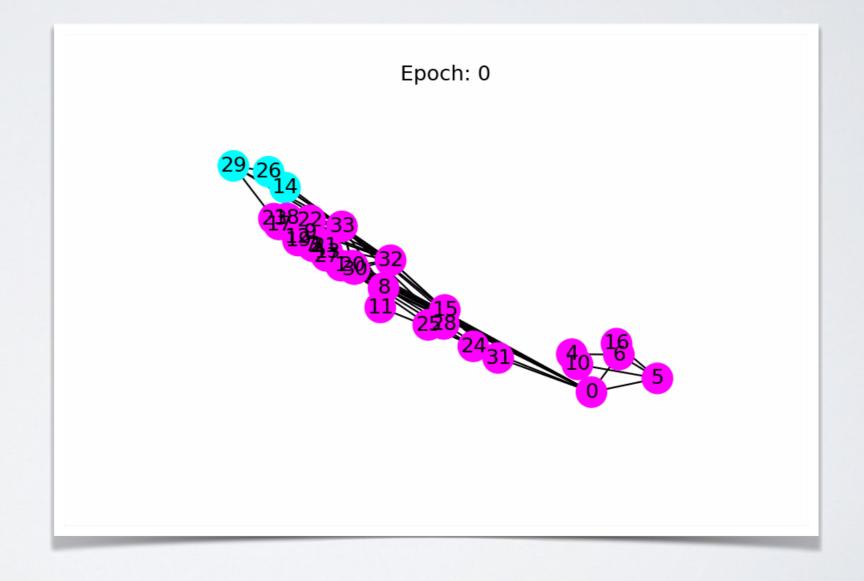
Step I: Combine one-hot to 5D



Result: Computed feature vector As expected, values for nodes 0 and 33 are opposed

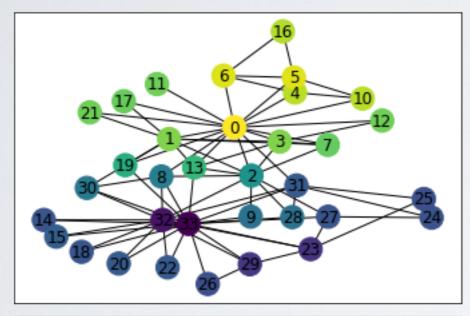
FITTING THE GCN

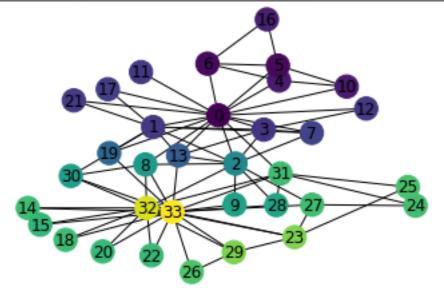
Epoch	0	Loss: 0.6987
Epoch	1	Loss: 0.6804
Epoch	2	Loss: 0.6634
Epoch	3	Loss: 0.6476
Epoch	4	Loss: 0.6326
Epoch	5	Loss: 0.6174
Epoch	6	Loss: 0.6017
Epoch	7	Loss: 0.5852
Epoch	8	Loss: 0.5684
Epoch	9	Loss: 0.5513
Epoch		Loss: 0.5338
Epoch		Loss: 0.5158
Epoch	12	Loss: 0.4976
Epoch	13	Loss: 0.4792
Epoch	14	Loss: 0.4605
Epoch	15	Loss: 0.4416
Epoch	16	Loss: 0.4225
Epoch	17	Loss: 0.4033
Epoch	18	Loss: 0.3842
Epoch	19	Loss: 0.3652
Epoch	20	Loss: 0.3464
Epoch	21	Loss: 0.3279
Epoch	22	Loss: 0.3096
Epoch	23	Loss: 0.2916
Epoch	24	Loss: 0.2741
Epoch	25	Loss: 0.2571
Epoch	26	Loss: 0.2407
Epoch	27	Loss: 0.2248
Epoch	28	Loss: 0.2095
Epoch	29	Loss: 0.1946
Epoch	30	Loss: 0.1803
Epoch	31	Loss: 0.1668
Epoch	32	Loss: 0.1541
Epoch	33	Loss: 0.1422
Epoch	34	Loss: 0.1312
Epoch	35	Loss: 0.1209
Epoch	36	Loss: 0.1113
Epoch	37	Loss: 0.1024
Epoch		Loss: 0.0940
Epoch	39	Loss: 0.0863
Epoch	40	Loss: 0.0793
Epoch	41	Loss: 0.0727
Epoch	42	Loss: 0.0667
Epoch	43	Loss: 0.0611
Epoch	44	Loss: 0.0560
Epoch	45	Loss: 0.0513
Epoch	46	Loss: 0.0470
Epoch	47	Loss: 0.0432
Epoch	48	Loss: 0.0396
Epoch	49	Loss: 0.0363
Epoch	50	Loss: 0.0333



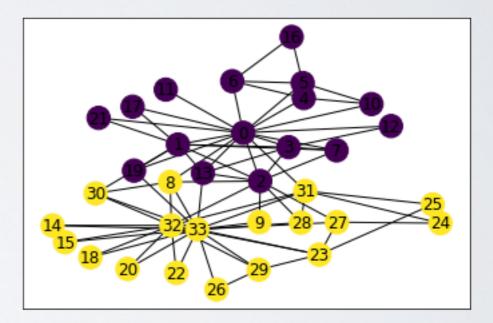
RESULTS

Features values





Highest feature as label



We retrieve the expected "communities"

GAT Graph ATtention networks

SELF-ATTENTION MECHANISM

- Mechanisms coming mostly from Language models
 - ▶ Transformers (as in GPT) are a particular type of self-attention

 In the normal GCN, a limit is the fix rule used to combine the neighbors attributes (weighted average)

$$h_i^{l+1} = \sum_{j \in N_i} \frac{1}{\sqrt{deg(i)}\sqrt{deg(j)}} h_j^l W^T$$

 Graph attention principle is to allow each node to "choose" what "attention" to give to each neighbor

$$h_i^{l+1} = \sum_{j \in N_i} \alpha_{ij} h_j^l W^T$$

- $lpha_{ij}$ attention from i to j

- Step I: a learnable attention matrix convert the node embeddings into new embeddings specific for Attention
 - $z_i = Wh_i$
 - W the learnable weights
 - h_i existing nodes features (embeddings/attributes)
- =>We don't want the node embedding to combine the meanings of:
 - The node position in the graph
 - The way it behaves in term of attention to others

- Step 2: concatenate both nodes embedding
 - $\bullet z_i | | z_j$
 - [a,b,c] || [f,e,d] => [a,b,c,f,e,d]
- =>To decide if a link is important, we will consider the attributes/embeddings of both nodes

- Step 2: Compute an attention coefficient a_{ij} using learnable weights
 - $e_{ij} = a^T[z_i | |z_j]$
 - *a* is a learnable vector, common to all node pairs, computing the attention from the combined embedding
 - > => Same principle as a perceptron/linear regression
 - > > Variant (as in transformer): compute $z_i \cdot z_j$
- Step 3: Add an activation function
 - $e_{ij} = ReLu(a^T[z_i | |z_j])$

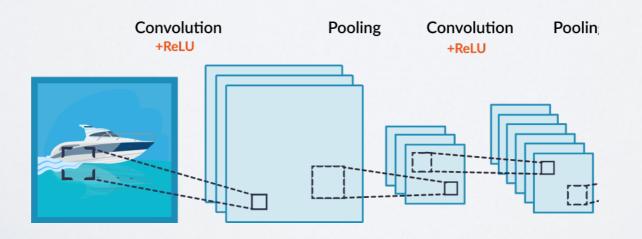
- Step 4: Softmax normalization
 - We have unnormalized attention score for each neighbor
 - Use Softmax to normalize attention
 - small value for a single node n, 0 value for all others => all attention on n !

$$\alpha_{ij} = softmax(e_{ij}) = \frac{exp(e_{ij})}{\sum_{k \in N_i} exp(e_{ik})}$$

- Finally: Multi-head attention
 - A single attention layer might not be powerful enough. What we described is called an **attention head**, and we typically have multiple heads
 - The resulting embeddings are then combined

Average (i.e.,
$$h_i = \frac{1}{n} \sum_{k=1}^n h_i^k$$
)

• Similar principle as the multiple channels of a convolution

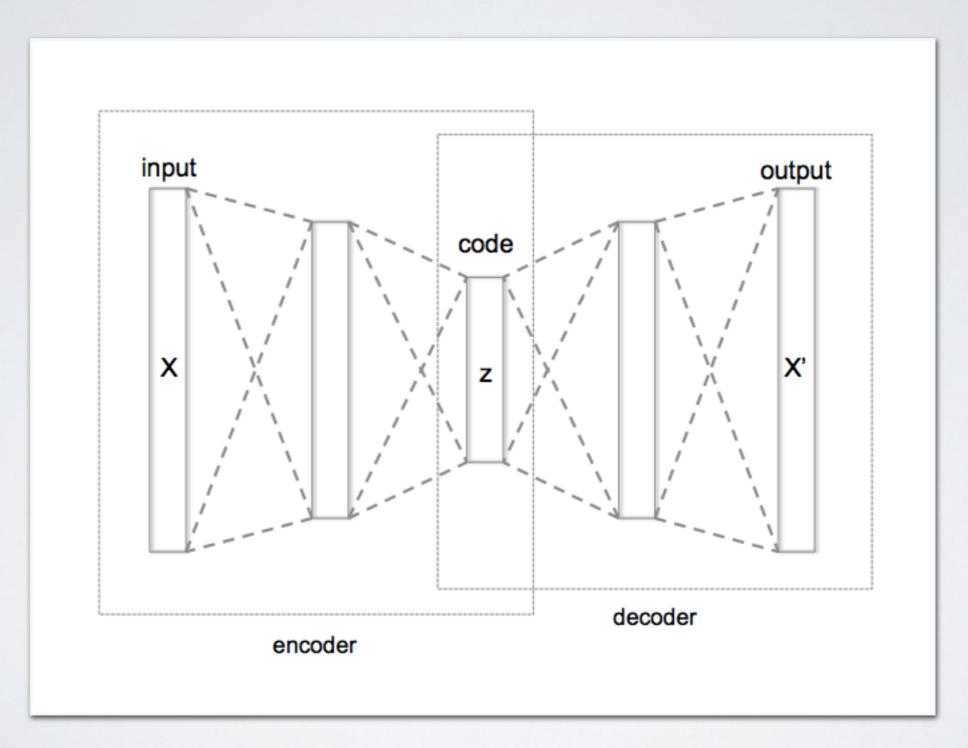


GRAPH AUTOENCODERS

AUTOENCODERS

- Autoencoders are mostly used for unsupervised learning using deep neural networks
- Typically, for images.
- Composed of two parts
 - An encoder
 - e.g., a classic sequence of convolutional layers
 - A decoder
 - e.g., an inverse architecture (e.g., the same layers in inverse order)
- In the middle is the "embedding", what we are interested in
 - Constrained to be small

AUTOENCODERS



AUTOENCODERS

• The objective is to

- Encode a complex object
 - e.g., a 3 color layers, 256 x 256 image
- Into a small-dimensional vector
 - e.g., vector of size | 28
- Such that these vectors allow to reproduce the output with minimal loss of information
- Many applications:
 - Visualization (like PCA)
 - Downstream task (these vectors can be used for classification, etc.)
 - Generate variations (Generative image models...)

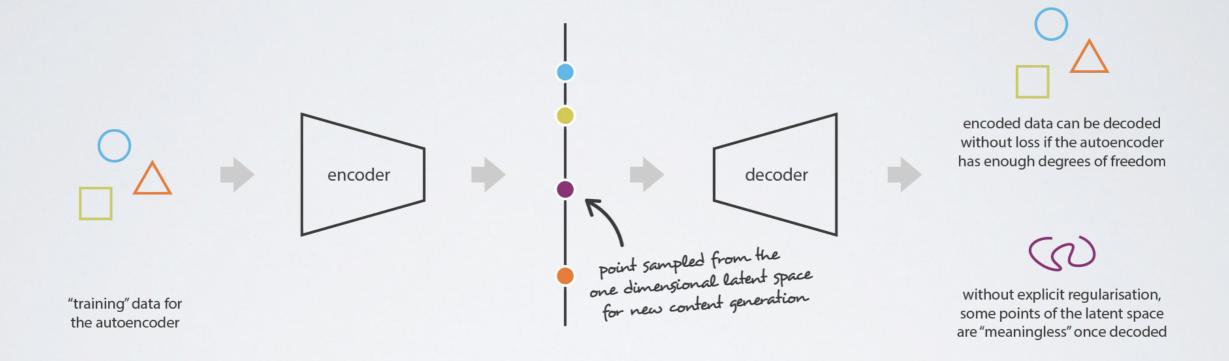
GRAPH AUTOENCODERS

- Same principle, but with graphs :)
- Classic architecture[1]:
 - Encoder: GCN layers (e.g., 2 layers)
 - Decoder: Dot product between embeddings (+activation)
 - Minimize the binary cross entropy between input and output adjacency matrices
- >=>Compute vectors for each node
 - such that their dot product is
 - Close to I if they are connected (parallel => similar vectors)
 - Close to 0 if they are not (orthogonal => different vectors)

VARIANT: VGAE

- VAE : Variational AutoEncoder
 - Popular improvement over classic AutoEncoder
- Limits of Autoencoders:
 - Embedding space is often poorly structured
 - Poor **continuity**: The "middle" vector between two vectors (v1,v2) do not correspond to a middle image between the two corresponding to v1/v2
 - Poor completeness: Space seems "sparse": many vectors correspond to nothing meaningful
- VAE solution:
 - Instead of encoding an input as a single point, we encode it as a distribution over the latent space

VAE



https://towardsdatascience.com/understanding-variational-autoencoders-vaes-f70510919f73



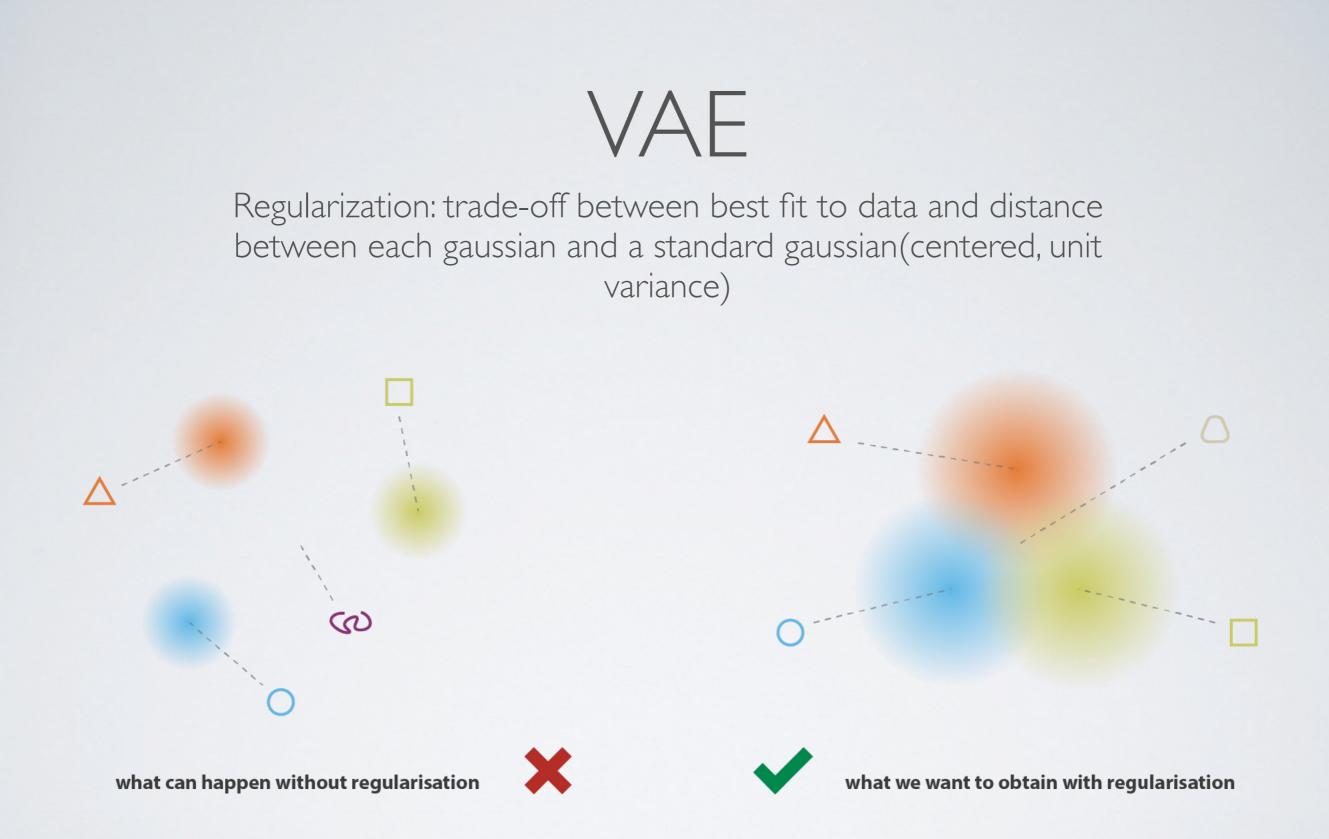


https://towardsdatascience.com/understanding-variational-autoencoders-vaes-f70510919f73

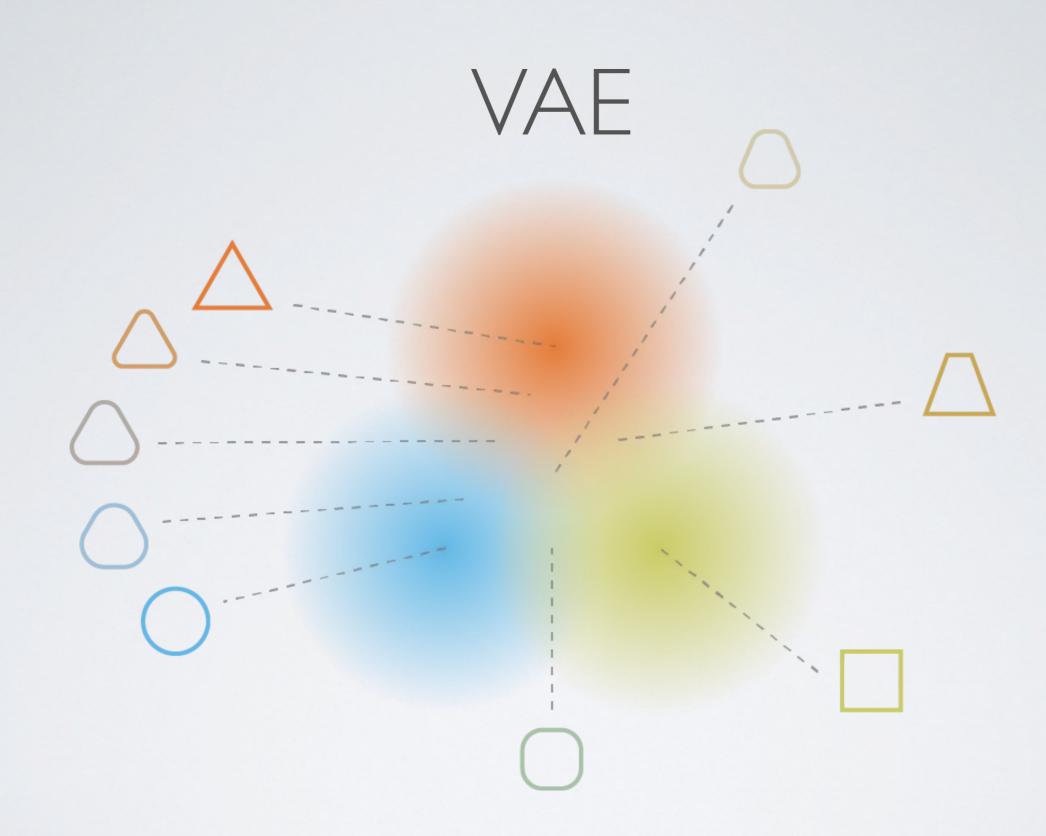
VAE

• The model is trained as follows:

- I) the input is encoded as gaussian **distribution** over the latent space
- 2) a point from the latent space is sampled from that distribution
- 3) the sampled point is decoded and the reconstruction error can be computed
- 4) finally, the reconstruction error is backpropagated through the network



https://towardsdatascience.com/understanding-variational-autoencoders-vaes-f70510919f73



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VGAE

- Simple adaptation to graphs, i.e., a classic graph autoencoder in which the encoding part is replaced by Variational mechanism.
- In practice:
 - Layer I: normal GCN
 - Layer 2: two parallel GCN layers
 - One to learn the centroid
 - One to learn the variance (diagonal of the covariance matrix)
 - =>For each node, instead of having I vector of size d, we have two vectors of size d
 - To decode, we take a random point from the multivariate gaussian

- Observed network: current state
- Link prediction: What edge
 - Might appear in the future (future link prediction)
 - Might have been missed (missing link prediction)
- Many applications
 - Recommender systems
 - Drug/healness prediction, ...

- Classification objective
 - Binary classes: edge/No edge
 - Usually, evaluation based on class probability
 AUC, AP...
- Evaluation process
 - Hide some of the edges in the graph
 - Check that
 - Training on the remaining edges
 - We predict well the removed ones

- Classic methods
 - Common Neighbors
 - Adamic Adar
 - ٠...
 - =>Work only on nodes at distance two
- Advanced methods
 - Graph embedding (DeepWalk, Node2Vec)
 - Use dot product of embedding as score, or other variants, e.g., training a classification on vectors
 - Community structure, random walks
 - > =>Do not take node features into account

- Using VGAE
 - The objetive of VGAE is to reconstruct the graph, i.e., to predict which edge is present or not =>Directly a link prediction objective
 - VGAE final step: dot product of embeddings
- Edge prediction score: result of the dot product of node vectors

- Using directly a GNN
 - GNNs produce node embeddings in the output
 - We need to combine node embeddings
- Two (main) solutions
 - Create a combined vector from two independent vectors, and add a linear layer for classification
 - Use directly a vector-to-scalar operation

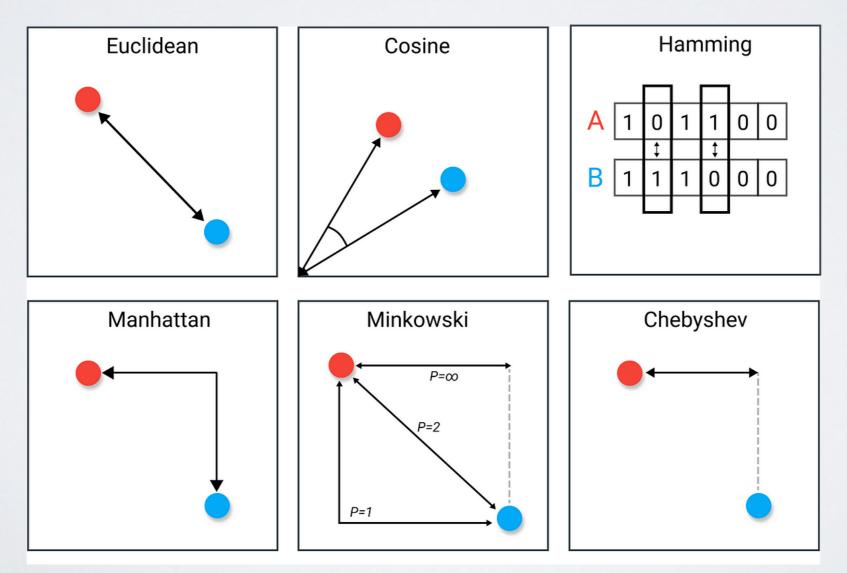
- Combining two node vectors into a node-pair vector
 - Vector concatenation [x1,x2] [x3,x4] = > [x1,x2,x3,x4]
 - LI difference [x1,x2][x3,x4]=>[x1-x3,x2-x4]

. . .

- Hadamard Product [x1,x2][x3,x4]=>[x1*x3,x2*x4]
- Followed by a classification task on this new vector

Combining two node vectors into a scalar

dot productpproxunnormalized cosine similarity



TRANSDUCTIVE / INDUCTIVE

Transductive

- What we discussed until now:
 - We have access to the whole graph at training time
 - We just don't see all the labels (test, prediction)
- Inductive
 - Train on a set of nodes/graphs
 - Results can be applied to unseen nodes/graphs
 - A GCN layer can be trained on multiple (sub)networks, and learned weights used on a new scenario (but not very efficient)
 - GraphSAGE=>Works for each node on a local graph centered on the node, by sampling a fixed number of neighbors.Transform the graph problem in a more classic problem.

MULTI-PARTITE GRAPHS

- Nodes of multiple types:
 - Items/Users
 - Drug/illness
 - ٠...
- Each type of node has their own attributes
 - Cannot learn a single GCN layer
- =>Learn 2 independent layers
 - User attributes to Item attributes
 - Item attributes to User attributes